Chaotic dynamics in Bertrand model with technological innovation

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Abstract. In this paper, the dynamics of a Bertrand duopoly game with technology innovation have been studied, which contains boundedly rational and naive players. They have been analyzed that the stability of the equilibrium point, the bifurcation and chaotic behavior of the dynamic system. It has been proved that technology innovation has played a very important role in the stability of Nash equilibrium point. Technology innovation can enlarge the stability region of the speed and control the chaos of the dynamic system effectively.

Keywords: Bertrand duopoly games, technology innovation, nash equilibrium point, chaos.

1. Introduction

Cournot game [1] and Bertrand game [2] are the classic models of oligopoly competition. In Cournot game, oligopoly enterprises put the product output as the decision variable and choose the optimal output for high profits. In Bertrand competition, duopoly enterprises play a game based on product differentiation and determine a proper price to obtain the huge profits. In reality, oligopoly enterprises make the output and price decision dynamically and often adjust the decision based on market demand, the decision of competitors, their own production capacity and so on.

In the past, a large number of literatures did the research on dynamic behavior of Cournot and Bertrand game, such as A.A. Elsadany [3], Xiaolong Zhu [4], H. N. Agiza [5], A. K. Naimzada [6], Jixiang Zhang [7], Luciano Fanti [8], Baogui Xin [9] and so on, which involved several adjustment rules: naïve [8, 10, 11], adaptive [5, 12], bounded rational [3, 4, 7, 13, 14] and local monopolistic approximation [3, 14]. Basically, research results of all these papers show that bifurcation and chaos exist in the dynamic system of Cournot and Bertrand game [3-14]. The parameter, adjustment speed of bounded rational player is an important factor which influences the stability of Nash equilibrium point and incurs bifurcation and chaos. What's more, keeping low adjustment speed can control the chaos.

Previous research conclusions seem unified but not rich. It still lacks studies about the influence of other parameters on the stability of dynamic system. It is well-known that technology innovation plays an important role in economic development. It can enhance the competitive advantage and increase profit of enterprises. Then, can it improve the stability of dynamic systems?

Following the method of Zhang [7] and Agiza [5], this article explores the dynamics of Bertrand duopoly game between boundedly rational and naive players and study the effects of technology innovation on the dynamics of Bertrand model. The paper is structured as follows: In Section 2, a dynamic Bertrand duopoly game model was built which is composed of players that produce heterogeneous products and have different price adjustment rules. In Section 3, the dynamic behaviors and the equilibrium points were studied. The conditions for the existence and local stability of the equilibrium points will be also analyzed. In Section 4, the dynamic system was simulated via many bifurcation figures. The Section 5 drew the conclusion.

2. The model

We consider a Bertrand-type duopoly market where two oligopolies choose different prices for their heterogeneous products. Players can decide the prices according to the adjustment rules.

Let $p_i(t)$, $i = 1, 2$ represents the price of firm i at discrete-time periods $t = 0, 1, 2, ...$, Q_i represents the output. Following Zhang [7], suppose the market demand function of the players is:

$$Q_i = a - 2bp_i + dp_j, \tag{1}$$

where $a > 0$, $b > 0$, $i, j = 1, 2$, $i \neq j$. The parameter d measures the degree of substitution of the two products. Large d represents big degree of substitution.

Positive parameter A_i is the initial marginal cost of firm i. The production cost will be reduced by technological innovation. Let x_i represents the reduction degree of the marginal cost of the firm i. x_i is positive correlation with technology innovation investment and can be used to measure the degree of technology innovation. Furthermore, the firm i can benefit from another firm's innovation because technique innovation has externality and spillover effect. Let $\beta \in [0,1]$ is the degree of technology spillover. The marginal cost function of the players can be assumed as follows:

$$c_i = A_i - \left(x_i + \beta x_j\right), \tag{2}$$

where, $c_i \geq 0$ namely $x_i + \beta x_j \leq A_i$ hold.

With these assumptions, the profit of the firm i in the single period can be given by:

$$\pi_i = (p_i - c_i)Q_i = \left[p_i - A_i + \left(x_i + \beta x_j\right)\right]\left(a - 2bp_i + dp_j\right). \tag{3}$$

From the profit maximization by player i, the marginal profits in period t are obtained as:

$$\frac{\partial \pi_i}{\partial p_i} = a - 2bp_i + dp_j + b\left[A_i - \left(x_i + \beta x_j\right)\right]. \tag{4}$$

Then, the optimal price response function of firm i can be given by:

$$p_i = \frac{a + dp_j + b[A_i - (x_i + \beta x_j)]}{2b}. \tag{5}$$

Information in the market usually is incomplete. Supposing players use different expectations to adjust the prices. Following Zhang [7]and Agiza [5], suppose player 1 is boundedly rational [7] and player 2 is naïve [5].

Boundedly rational player 1 makes its price decision based on an estimate of the marginal profit $\partial\pi_1/\partial p_1$ [7]. Namely it decides to increase its price p_1 if it has a positive marginal profit, or decreases its price when the marginal profit is negative. Then the dynamical equation of player 1 can be given by:

$$p_1(t+1) = p_1(t) + kp_1(t)\frac{\partial \pi_1}{\partial p_1(t)}, \tag{6}$$

where k is a positive parameter which reflects the speed of price adjustment.

Naive player 2 makes its price decision according to the naive expectations rule [8]. The player 2 decides its prices with his reaction function $(a + dp_1 + b[A_2 - (x_2 + \beta x_1)])/2b$. Hence the dynamic equation of the naive expectation player 2 can be given by:

$$p_2(t+1) = \frac{a + dp_1(t) + b[A_2 - (x_2 + \beta x_1)]}{2b}. \quad (7)$$

With above assumptions, the duopoly game with heterogeneous players is formed from combining Eqs. (6) and (7). Then the dynamical system of the heterogenous players is described as:

$$\begin{cases} p_1(t+1) = p_1(t) + kp_1(t)[a - 2bp_1(t) + dp_2(t) + b(A_1 - x_1 - \beta x_2)], \\ p_2(t+1) = \dfrac{a + dp_1(t) + b(A_2 - x_2 - \beta x_1)}{2b}. \end{cases} \quad (8)$$

3. Nash equilibrium and local stability

In this part the equilibria points of dynamic system will be first studied Eq. (8), and then the stability will be discussed.

The dynamic duopoly game will achieve a Nash Equilibrium at last. The possible equilibrium point of the map Eq. (8) can be obtained as nonnegative solution of the nonlinear algebraic system:

$$\begin{cases} p_1[a - 2bp_1 + dp_2 + b(A_1 - x_1 - \beta x_2)] = 0, \\ p_2 = \dfrac{a + dp_1 + b(A_2 - x_2 - \beta x_1)]}{2b}. \end{cases} \quad (9)$$

Find that the system (9) is not associated with the parameter k. After the calculation of the system it was found that the map has two equilibrium points:

$$E_1 = (0, {p_2}^0), \;\; E_2 = ({p_1}^*, {p_2}^*), \quad (10)$$

where:

$$\begin{aligned} {p_2}^0 &= \frac{a + b[A_2 - x_2 - \beta x_1]}{2b}, \\ {p_1}^* &= \frac{2b[a + b(A_1 - x_1 - \beta x_2)] + d[a + b(A_2 - x_2 - \beta x_1)]}{4b^2 - d^2}, \\ {p_2}^* &= \frac{2b[a + b(A_2 - x_2 - \beta x_1)] + d[a + b(A_1 - x_1 - \beta x_2)]}{4b^2 - d^2}. \end{aligned} \quad (11)$$

In the traditional economic view, non-negative equilibrium is meaningful. Obviously, E_1 is a boundary equilibria (${p_2}^0 > 0$). E_2 is the unique Nash equilibrium point and has economic meaning provided that:

$$\begin{cases} 2b[a + b(A_1 - x_1 - \beta x_2)] + d[a + b(A_2 - x_2 - \beta x_1)] > 0, \\ 2b[a + b(A_2 - x_2 - \beta x_1)] + d[a + b(A_1 - x_1 - \beta x_2)] > 0, \\ 2b > d, \end{cases} \quad (12)$$

where, the above two inequalities are obvious, then Eq. (12) is equivalent to $2b > d$.

In order to study the local stability of equilibrium, the Jacobian matrix of map Eq. (8) should be considered. The matrix form is as follows:

$$J(E) = \begin{bmatrix} 1 + k[a - 4bp_1 + dp_2 + b(A_1 - x_1 - \beta x_2)] & dkp_1 \\ \dfrac{d}{2b} & 0 \end{bmatrix}. \quad (13)$$

The equilibrium point is stable only when all eigenvalues λ_i $(i = 1, 2)$ of the Jacobian matrix satisfy $|\lambda_i| < 0$. According to this theory, the following result about E_1 can be received.

Proposition 1. The equilibrium point E_1 of system Eq. (8) is a saddle point.

Proof. The Jacobian matrix of E_1 has the form:

$$J(E_1) = \begin{bmatrix} 1 + k[a + dp_2{}^0 + b(A_1 - x_1 - \beta x_2)] & 0 \\ \frac{d}{2b} & 0 \end{bmatrix}. \tag{14}$$

Its' eigenvalues are:

$$\lambda_1 = 1 + k[a + dp_2{}^0 + b(A_1 - x_1 - \beta x_2)], \ \ \lambda_2 = 0.$$

For the condition that a, b, d, k, c_i are all positive parameters, $|\lambda_1| > 1$ is workable. Then the equilibrium point E_1 is a saddle node. The proof of the proposition is completed.

Next the local stability of the Nash equilibrium point E_2 will be studied. The Jacobian matrix of E_2 is:

$$J(E_2) = \begin{bmatrix} 1 + k[a - 4bp_1{}^* + dp_2{}^* + b(A_1 - x_1 - \beta x_2)] & dkp_1{}^* \\ \frac{d}{2b} & 0 \end{bmatrix}, \tag{15}$$

where, the trace of $J(E_2)$ is:

$$T = Tr\big(J(E_2)\big) = 1 + k[a - 4bp_1{}^* + dp_2{}^* + b(A_1 - x_1 - \beta x_2)]. \tag{16}$$

The determinant of $J(E_2)$ is:

$$D = Det\big(J(E_2)\big) = -\frac{d^2kp_1{}^*}{2b}. \tag{17}$$

The characteristic equation of $J(E_2)$ is:

$$P(\lambda) = \lambda^2 - T\lambda + D = 0. \tag{18}$$

The discriminant is:

$$\Delta = T^2 - 4D. \tag{19}$$

Since $\Delta = T^2 + 2d^2kp_1{}^*/b > 0$, the eigenvalues of Nash equilibrium E_2 are real.

Necessary and sufficient conditions for local stability of the Nash equilibrium E_2 are the Jury's condition, which is given by:

$$\begin{cases} \text{(I)}: 1 + T + D > 0, \\ \text{(II)}: 1 - T + D > 0, \\ \text{(III)}: 1 - D > 0. \end{cases} \tag{20}$$

Since:

$$1 - T + D = -k[a - 4bp_1{}^* + dp_2{}^* + b(A_1 - x_1 - \beta x_2)] - \frac{d^2kp_1{}^*}{2b}. \tag{21}$$

Then replace p_1^*, p_2^* Eqs. (21) can be simplified as:

$$\frac{2kb[a+b(A_1-x_1-\beta x_2)]+kd[a+b(A_2-x_2-\beta x_1)]}{2b}>0,$$

(II) is always satisfied.

Since:

$$1-D=1+\frac{d^2kp_1^*}{2b}>0, \tag{22}$$

(III) is always satisfied.

Then focus on the inequality (I).

Since:

$$1+T+D=2+k[a+b(A_1-x_1-\beta x_2]-\frac{k(8b^2+d^2)p_1^*}{2b}+kdp_2^*. \tag{23}$$

Then replace p_1^*, p_2^*. Since, $2b>d$ (I) is equivalent to:

$$4b(4b^2-d^2)-k(4b^2+d^2)[2b(a+bc_1)+d(a+bc_2)]>0. \tag{24}$$

From what has been mentioned above, the following conclusion can come out:

Proposition 2. The Nash equilibrium at E_2 is stable if and only if the inequality Eq. (24) holds.

Proposition 2 characterizes the stability region in which the Nash equilibrium E_2 is local stable. The violation of the inequality Eq. (24) will lead to a flip bifurcation [3].

Noticing that the stability region is associated with x_i and β. The propositions can be given about the degree of technology innovation x_i and the degree of technology spillover β.

Proposition 3. When $x_1>x_1{}^0$, the evolution of price system Eq. (8) is in a stable state and E_2 is the Nash equilibrium point. Otherwise, the price evolution is in bifurcation or chaos. Where:

$$x_1{}^0=\frac{2b[a+b(A_1-\beta x_2)]+d[a+b(A_2-x_2)]-\dfrac{4b(4b^2-d^2)}{k(4b^2+d^2)}}{2b^2+db\beta}. \tag{25}$$

Proof. According to the stability theory of Jury's condition, the flip bifurcation occurs when $1+T+D=0$. Namely:

$$4b(4b^2-d^2)-k(4b^2+d^2)[2b(a+bc_1)+d(a+bc_2)]=0. \tag{26}$$

Then $x_1=x_1{}^0$.

So, the system is in stable when $x_1>x_1{}^0$, otherwise in bifurcation or chaos.

Proposition 4. When $x_2>x_2{}^0$, the evolution of price system Eq. (8) is in a stable state and E_2 is the Nash equilibrium point. Otherwise, the price evolution is in bifurcation or chaos. Where:

$$x_2{}^0>\frac{2b[a+b(A_1-x_1)]+d[a+b(A_2-\beta x_1)]-\dfrac{4b(4b^2-d^2)}{k(4b^2+d^2)}}{2b^2\beta+db}. \tag{27}$$

Proof. According to the stability theory of Jury's condition, the flip bifurcation occurs when $1+T+D=0$. Namely:

$$4b(4b^2-d^2)-k(4b^2+d^2)[2b(a+bc_1)+d(a+bc_2)]=0. \tag{28}$$

Then $x_2 = {x_2}^0$.

So, the system is in stable when $x_2 > {x_2}^0$, otherwise in bifurcation or chaos.

From the above description and Proposition, it can be concluded that high technology innovation is beneficial to obtain a steady state and the Nash equilibrium profit. It can expand the stable region and enhance the stability of the product price of market to increase technical innovation.

4. Numerical simulations

The purpose of this part is to illustrate the qualitative behavior of the solutions of the duopoly dynamic system Eq. (8) and provide some numerical evidences to prove above results.

In Fig. 1 ($a = 5$, $b = 1$, $d = 0.3$, $A_1 = 2$, $A_2 = 3$, $x_2 = 2$, $\beta = 0.5$, $k = 0.32$) Nash equilibrium is locally stable approaches to the stable point $({p_1}^*, {p_2}^*) = (3.031, 3.229)$ for large values of x_1, to be specific, when $x_1 > 0.905$. With the reduction of x_1, the Nash equilibrium point becoming instable, period-halving bifurcation and chaos will occur.

In Fig. 2 ($a = 5$, $b = 1$, $d = 0.3$, $A_1 = 2$, $A_2 = 3$, $x_1 = 1$, $\beta = 0.5$, $k = 0.32$) the Nash equilibrium is locally stable only when $x_2 > 1.769$ ($a = 5$, $b = 1$, $d = 0.3$, $A_1 = 2$, $A_2 = 3$, $x_1 = 1$, $\beta = 0.5$, $k = 0.32$). The dynamic system is in bifurcation or chaos if the technology innovation degree x_2 is small.

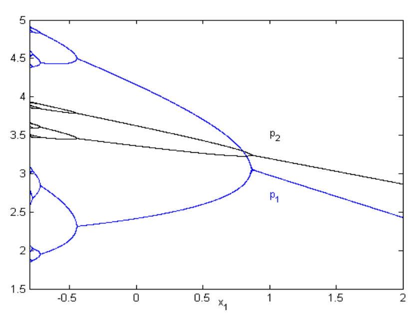

Fig. 1. Bifurcation diagram with respect to x_1

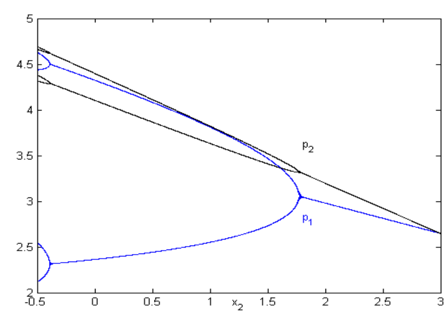

Fig. 2. Bifurcation diagram with respect to x_2

5. Conclusions

This paper established the price dynamic game model and then analyzed the influence of technology innovation on the equilibrium stability. The results show that technology innovation plays an important role in improving the stability of equilibrium. Specifically, it can enlarge the stability region and make the original bifurcation and chaos change into stability to increase the degree of technology innovation.

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