# Strength Prediction Model of Eutectic Composite Ceramics Mainly Composed of Rod-Shaped Crystals

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**Abstract.** In this paper, the effect of preexisting defects on strength of eutectic composite ceramic mainly composed of rod-Shaped crystals is considered. In line with the microstructure characteristic of eutectic composite ceramic, its micromechanical model is set up. Defects are assumed to be lamellar, and matrix around is transversely isotropic. Then, the damage stress field is obtained by Chaboche's damage theory. Finally, the micromechanical strength of eutectic ceramic composite is predicted, and its influencing factor is analyzed.

## 1. Introduction

During the sintering process, ceramic materials always experience complex physical and chemical reactions and temperature variation process, micro-pores and micro-cracks is inevitable, and the inherent brittleness of ceramic materials makes the reduction of strength by defects extremely notable. There were mainly two types of defects in ceramic materials: one is the micro-pore which has blunt boundary and located at boundary of interphase and matrix, the other is micro crack which is caused by phase transformation or thermal strain mismatch and located in matrix or triradius region. Most strength analysis to reinforced composite are based on the presumption of non-defects in materials, or assuming matrix is isotropic when to analyze the influence of defects [1, 2]. However, sizes of defects and micro-structures almost have the same order of magnitude, matrix around defects is anisotropic for some kinds of eutectic composite ceramics therefore. For defects in anisotropic matrix, crack tip stress field and failure mechanism is more complex. However, research has made considerable progress. For considerable size of the defect, the affect of lamellar defects is much stronger than spheroidal defects. We only analyze the lamellar defects in this article. Based on the microstructure of eutectic composite, defects is simplified as elliptical crack in anisotropic matrix, considering the influence to micro-stress field of defects, damage variable was obtained with the help of the interaction direct derivative (IDD) estimate [3] to analyze of the effective stress field. According to eigenstrain theory, and Griffith's fracture criterion, strength model is established.

Effective stress field for three-phase model

Eutectic composite mainly consist of rod-shaped crystals, particles and little defects dispersed in it. Assuming eutectic composite with defects consist of three-phase cell randomly distributed spatial with appropriate volume fraction of matrix, all of these cells have the same shape, and embedded into the equivalent medium.

In the three-phase cell, effective matrix around elliptical defect consist of the uniform distributed rod-shaped crystals, it is transversely isotropic ($x_1$ and $x_3$ compose the basal plane), and ellipsoid defect defined by

$$\left(\frac{X}{a_1}\right)^2+\left(\frac{Y}{a_2}\right)^2+\left(\frac{Z}{a_3}\right)^2\le 1 \qquad (1)$$

Where $a_3 << a_1, a_2$ , and it is perpendicular to the rod-shaped crystal.

The above-mentioned three-phase cell placed in the equivalent matrix, incremental compliance

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obtained by the IDD method [3]:

$$\boldsymbol{H} = \left(\boldsymbol{I} - \boldsymbol{H}^d \boldsymbol{\Omega}_p\right)^{-1} \boldsymbol{H}^d \tag{2}$$

where,

$$\boldsymbol{H}^d = \sum f_p \left(\boldsymbol{I} + \boldsymbol{\Omega}_p \boldsymbol{H}_p\right)^{-1} \boldsymbol{H}_p \tag{3}$$

$$\boldsymbol{\Omega}_p = \boldsymbol{C}_0 \left(\boldsymbol{I} - \boldsymbol{M}_p\right) \tag{4}$$

In the above equation, stiffness tensor $\boldsymbol{C}_0$ of matrix composed of rod-shaped crystals can reference Li et al. 2011 [4], and $\boldsymbol{M}_p$ is the corresponding Eshelby tensor, for anisotropic matrix, there's no analytical expression, numerical solution by Tian and Hu [5] as follows

$$M_{ijkl} = P_{ijpq} C^0_{pqkl} \tag{5}$$

where, $i, j, p, q, k, l \in (1,2,3)$,

$$P_{ijpq} = \frac{1}{16\pi} \int_{-1}^{1} dx \int_0^{2\pi} R d\phi \tag{6}$$

$$R = G^{-1}_{ip} K_q K_j + G^{-1}_{jp} K_p K_i + G^{-1}_{jp} K_q K_i + G^{-1}_{iq} K_p K_j \tag{7}$$

$$G_{ij} = G_{ji} = C^0_{ipjq} K_p K_q \tag{8}$$

and

$$K_1 = \sqrt{1 - x^2} \cos\varphi, \quad K_2 = \frac{a_1}{a_2} \sqrt{1 - x^2} \sin\varphi, \quad K_3 = \frac{a_1}{a_3} x \tag{9}$$

Each component of tensor $\boldsymbol{P}$ computed by Gaussian quadrature method. When the defect is strongly oblate, The components of Eshelby tensor have simple forms as

$$M^P_{31} = \frac{C^0_{13}}{C^0_{33}}, \; M^P_{32} = \frac{C^0_{23}}{C^0_{33}}, \; M^P_{33} = M^P_{44} = M^P_{55} = 1, \text{ The rest components are zero} \tag{10}$$

where $C^0_{11}$, $C^0_{12}$, $C^0_{13}$ are component of matrix stiffness tensor, and it is obvious that Eshelby tensor of crack is unrelated to the ratio of $a_2 / a_3$ [6].

Assume that all defects for the same shape with stiffness of zero, that

$$\boldsymbol{H}^d = f_p \boldsymbol{\Omega}_p^{-1} \tag{11}$$

so

$$\boldsymbol{H} = \left(\boldsymbol{I} - \boldsymbol{H}^d \boldsymbol{\Omega}\right)^{-1} \boldsymbol{H}^d = \left(f_p^{-1} - 1\right)^{-1} \left[\boldsymbol{C}_0 \left(\boldsymbol{I} - \boldsymbol{M}_p\right)\right]^{-1} \tag{12}$$

Effective stiffness $\boldsymbol{C}_e$ of cell with defect denoted by

$$\boldsymbol{C}_e = \left(\boldsymbol{C}_0^{-1} + \boldsymbol{H}\right)^{-1} \tag{13}$$

Considering the spatial randomness of cell distribution, the macroscopic equivalent stiffness is denoted by reference [7]

$$\boldsymbol{C}_p^E = \begin{bmatrix} C_{11} & C_{12} & C_{12} & & & \\ C_{12} & C_{11} & C_{12} & & 0 & \\ C_{12} & C_{12} & C_{11} & & & \\ & & & \frac{C_{11}-C_{12}}{2} & & \\ & & & & \frac{C_{11}-C_{12}}{2} & \\ & 0 & & & & \frac{C_{11}-C_{12}}{2} \end{bmatrix} \quad (14)$$

where

$$\begin{aligned} C_{11} &= \frac{1}{5}(C_{11}^e + C_{22}^e + C_{33}^e) + \frac{2}{15}(C_{12}^e + C_{13}^e + C_{23}^e) + \frac{4}{15}(C_{44}^e + C_{55}^e + C_{66}^e) \\ C_{12} &= \frac{1}{15}(C_{11}^e + C_{22}^e + C_{33}^e) + \frac{4}{15}(C_{12}^e + C_{13}^e + C_{23}^e) - \frac{2}{15}(C_{44}^e + C_{55}^e + C_{66}^e) \end{aligned} \quad (15)$$

Here $C_{ij}^e$ are components of $\boldsymbol{C}_e$ when $f_p = 0$, effective tensor $\boldsymbol{C}_0^E$ of material without defects obtained.

According to the Chaboche's damage theories, define the fourth-order damage variable $\boldsymbol{D}$

$$\boldsymbol{D} = \boldsymbol{I} - \tilde{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^{-1} \quad (16)$$

in which

$$\boldsymbol{\Lambda} = \boldsymbol{C}_0^E, \quad \tilde{\boldsymbol{\Lambda}} = \boldsymbol{C}_p^E \quad (17)$$

hence

$$\boldsymbol{D} = \boldsymbol{I} - \boldsymbol{C}_p^E \left(\boldsymbol{C}_0^E\right)^{-1} \quad (18)$$

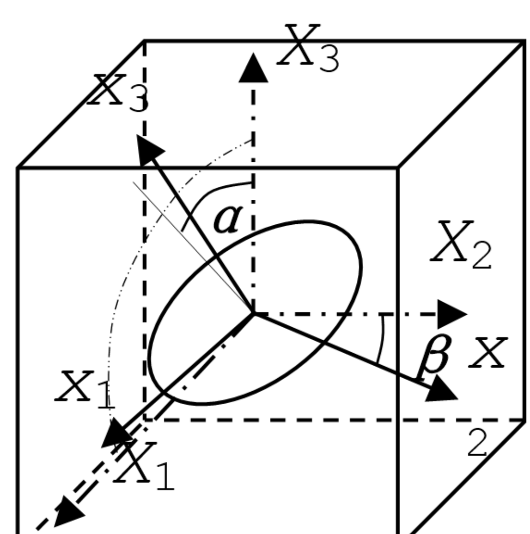


**Figure 1.** Relationship between local coordinates and global coordinates.

Embedded the three-phase cell into effective medium mentioned above, considering the spatial randomness of cell distribution, the spatial relationships between local coordinate system and the global coordinate system shows in Fig. 1.

Suppose material under the simple tension of $\sigma_\infty$ in $X_3$ direction, effective stress on cell in the local coordinate system as

$$\boldsymbol{\sigma}_{kl}^{D} = g_{ki} g_{lj} \boldsymbol{\sigma}'_{ij} \tag{19}$$

where, $\boldsymbol{g} = \begin{bmatrix} \cos\alpha & -\sin\alpha\sin\beta & -\sin\alpha\cos\beta \\ 0 & \cos b & -\sin b \\ \sin\alpha & \cos\alpha\sin\beta & \cos\alpha\cos\beta \end{bmatrix}$.

$$\boldsymbol{\sigma}' = \left(\boldsymbol{I} - \boldsymbol{D}\right)^{-1} \boldsymbol{\sigma}_\infty = \boldsymbol{C}_0^E \left(\boldsymbol{C}_p^E\right)^{-1} \boldsymbol{\sigma}_\infty \tag{20}$$

$$\boldsymbol{\sigma}_\infty^T = \begin{bmatrix} 0 & 0 & \sigma_\infty & 0 & 0 & 0 \end{bmatrix} \tag{21}$$

In which, the superscript $T$ means transpose, therefore, the non-zero components are

$$\begin{aligned} \sigma_{33}^D &= \sigma_\infty B_{33} = \sigma_\infty \left(A_{1133}\sin^2\alpha\cos^2\beta + A_{2233}\sin^2\alpha\sin^2\beta + A_{3333}\cos^2\alpha\right) \\ \sigma_{31}^D &= \sigma_\infty B_{31} = \sigma_\infty \left(A_{1133}\cos^2\beta + A_{2233}\sin^2\beta + A_{3333}\cos^2\alpha\right)\sin\alpha\cos\beta \\ \sigma_{32}^D &= \sigma_\infty B_{32} = \sigma_\infty \left(A_{2233} - A_{1133}\right)\sin\beta\cos\beta\cos\alpha \end{aligned} \tag{22}$$

strength prediction

For each cell, let $\sigma^D$ be the effective applied stress at infinity. The boundary conditions on the crack surface are

$$\sigma_{3j}^D + \sigma_{3j} = 0, \quad j = 1,2,3 \tag{23}$$

here $\sigma_{3j}$ is the stress disturbance due to the crack. This stress disturbance is simulated by the eigenstress derived from an equivalent ellipsoidal inclusion with eigenstrain $\varepsilon^*$ [8] and

$$\sigma_{ij} = C_{ijkl}^0 \left[\frac{a_1 a_2 a_3}{4\pi} C_{pqmn}^0 \varepsilon_{mn}^* \int_{s^2} G_{kplq}\left(x\right) u^{-3} dS\left(x\right) - \varepsilon_{kl}^*\right] \tag{24}$$

where $C_{pqmn}^0$ is corresponding component of fourth-order stiffness tensor of matrix.

$u = \left(a_1^2 x_1^2 + a_2^2 x_2^2 + a_3^2 x_3^2\right)^{\frac{1}{2}}$, $G_{kplq}\left(x\right) = x_l x_q K_{kp}^{-1}$, $K_{kp} = C_{pqmn}^0 x_i x_j$. For the following variable substitution

$$x_1 = \sqrt{1-t^2}\cos\varphi, \quad x_2 = \sqrt{1-t^2}\sin\varphi, \quad x_3 = t \tag{25}$$

$G_{kplq}$ is expressed as follows

$$G_{kplq}\left(\theta, t\right) = G_{kplq}\left[\sqrt{1-x_3^3}\cos\theta, \sqrt{1-x_3^3}\sin\theta, x_3\right] \tag{26}$$

$$dS\left(x\right) = dx_3 d\theta = \left(1-x_3^2\right)^{3/2} d\theta \tag{27}$$

thus, we have

$$\int_{S^2} G_{kplq}(x)u^{-3}dS(x) = \int_0^{2\pi} \frac{d\theta}{a_1^2a_2^2a_3^2}\int_{-\infty}^{\infty} \frac{G_{kplq}(\cos\theta, \sin\theta, t)}{\left(\cos^2\theta / a_2^2a_3^2 \sin^2\theta / a_1^2a_3^2 + t^2 / a_1^2a_2^2\right)^{3/2}}dt \tag{28}$$

Eq. (24) can be simplified as

$$\sigma_{ij} = \frac{1}{4\pi}C^0_{ijkl}C^0_{pqmn}\varepsilon^*_{mn}\left[4\pi G_{kplq}(0,0,1) - a_3\Pi_{kplq}\right] - C^0_{ijkl}\varepsilon^*_{mn} = \frac{1}{4\pi}a_3C^0_{ijkl}C^0_{pqmn}\varepsilon^*_{mn}\Pi_{kplq} \tag{29}$$

as $a_3 \to 0$, substituting Eq. (29) into Eq. (23), we have

$$\sigma^D_{3j} + a_3L_{ijmn}\varepsilon^*_{mn} = 0 \tag{30}$$

Here,

$$L_{ijmn} = -\frac{1}{4\pi}C^0_{ijkl}C^0_{pqmn}\Pi_{kplq} \tag{31}$$

The interaction energy is

$$\Delta W = -\frac{1}{2}\frac{4\pi}{3}a_1a_2a_3\sigma_D\varepsilon^* = -\frac{2\pi}{3}a_1a_2a_3\sigma^D_{3j}\varepsilon^*_{3j} \tag{32}$$

Suppose the defects propagate self-similarly, the Griffith fracture criterion is

$$dJ = 0 \tag{33}$$

where

$$J = \Delta W + 2\pi a_1a_2\gamma = 0 \tag{34}$$

where $\gamma$ is the fracture surface energy of the material. By substituting Eq. (32) and Eq. (34) into Eq. (33), we have

$$\left[\gamma - \frac{(\sigma^D_{33})^2P_{3333}}{L^2_{3333}} - \frac{(\sigma^D_{32})^2P_{3232}}{L^2_{3232}} - \frac{(\sigma^D_{31})^2P_{3131}}{L^2_{3131}}\right]\frac{\delta a_1}{a_1} = 0 \tag{35}$$

or

$$\left[\gamma - \frac{(\sigma^D_{33})^2Q_{3333}}{L^2_{3333}} - \frac{(\sigma^D_{32})^2Q_{3232}}{L^2_{3232}} - \frac{(\sigma^D_{31})^2Q_{3131}}{L^2_{3131}}\right]\frac{\delta a_2}{a_2} = 0 \tag{36}$$

Hence the micromechanical strength is

$$\sigma_c = \left(\frac{\gamma}{\frac{B^2_{33}P_{3333}}{L^2_{3333}} + \frac{B^2_{32}P_{3232}}{L^2_{3232}} + \frac{B^2_{31}P_{3131}}{L^2_{3131}}}\right)^{\frac{1}{2}} \tag{37}$$

or

$$\sigma_c = \left( \frac{\gamma}{\frac{B_{33}^2 Q_{3333}}{L_{3333}^2} + \frac{B_{32}^2 Q_{3232}}{L_{3232}^2} + \frac{B_{31}^2 Q_{3131}}{L_{3131}^2}} \right)^{\frac{1}{2}} \tag{38}$$

## 2. Conclusions

Assuming material is completely brittle, based on Griffith fracture theory, crucial strength model containing lamellar defects is obtained, the spatial randomness distribution of defect is take into consideration, and analyzed the strength of composite material under the conditions of simple tension, the defect characteristics to strength is analyzed.

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