830. Nonlinear dynamic characteristics of SMA simply supported beam in axial stochastic excitation

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Abstract. In this paper, nonlinear dynamic characteristics of shape memory alloy (SMA) simply supported beam in axial stochastic excitation were studied. Von del Pol nonlinear difference item was introduced to interpret the hysteresis phenomenon of the strain-stress curve of SMA, and the hysteretic nonlinear dynamic model of SMA simply supported beam in axial stochastic excitation was developed. The local stochastic stability of the system was analyzed according to the largest Lyapunov exponent, and the global stochastic stability of the system was discussed in singular boundary theory. The steady-state probability density function and the joint probability density function of the system were obtained in quasi-nonintegrable Hamiltonian system theory. The result of simulation shows that the stability of the trivial solution varies with bifurcation parameter, and stochastic Hopf bifurcation appears in the process. The result is helpful to stochastic bifurcation control to SMA simply supported beam.

Keywords: shape memory alloy (SMA), hysteretic nonlinearity, stochastic bifurcation.

Introduction

Shape Memory Alloy (SMA) is a kind of smart materials and applied in engineering field widely. It has many special characteristics such as shape memory effect, large damping and super-elasticity, based on which the SMA smart structure can be designed to reduce engineering vibration. SMA simply supported beam is a kind of basic smart structure, which was applied in vibration reduction field widely and has complex nonlinear dynamical characteristics. Lau analyzed vibration characteristics of SMA beams with different boundary conditions [1]. Liew studied the pseudoelastic behavior of a SMA beam by the element-free Galerkin method [2]. Zbiciak discussed dynamic characteristics of pseudoelastic SMA beam [3]. Scarpa developed spectral element formulation for SMA beams under random vibration excitation [4]. Hashemi developed the dynamic model of SMA beam [5]. Collet analyzed vibration behavior of SMA beam under dynamical loading [6].

This paper aims to offer a kind of analysis method to nonlinear dynamical characteristic of SMA simply supported beam in axial stochastic excitation in theoretically. Von del Pol nonlinear difference item was introduced to interpret the hysteresis phenomenon of strain-stress curve of SMA, and the hysteretic nonlinear dynamic model of SMA simply supported beam in axial stochastic excitation was developed. The local stochastic stability of the system was analyzed according to the largest Lyapunov exponent, and the global stochastic stability of the system was discussed in singular boundary theory. The steady-state probability density function and the joint probability density function were obtained in quasi-nonintegrable Hamiltonian system theory. Finally, the theoretic result was proved by simulation.

Hysteresis Nonlinear Model of SMA Simply Supported Beam in Axial Stochastic Excitation

The strain-stress curve of SMA was shown in Fig. 1. Obviously, there is hysteretic nonlinearity in the strain-stress curve of SMA. Most of SMA models were based on thermodynamics theory and micromechanics theory, where the percentage content of martensite was taken as main variable of stress-strain equation. As results, those SMA models were mostly shown as equations with subsection function or double integral function, and hard to be analyzed in theory [7-13]. Usually, research results to those models can only be obtained by numerical method or experiment method [14-18]. In this paper, Von del Pol hysteretic cycle model was introduced to describe the hysteretic nonlinear characteristic of SMA.

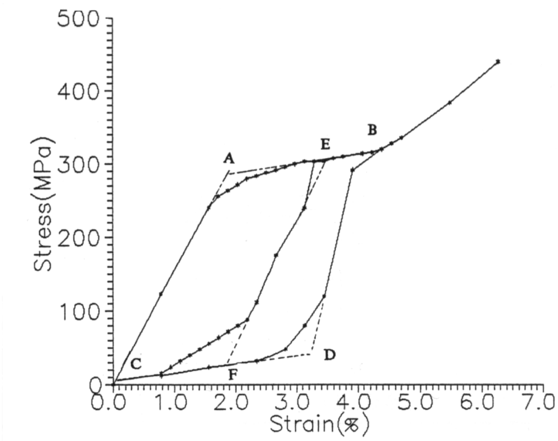

Fig. 1. The strain-stress curve of SMA

The initial Von del Pol hysteretic model describes hysteretic loop which is symmetrical about the initial point (0, 0). It can be shown as follows:

$$y = f(x) = f_0(x) + a\left[1 - \left(\frac{x}{b}\right)^2\right]\dot{x} \tag{1}$$

where $f_0(x)$ is skeleton curve of hysteretic loop and usually expressed in polynomial function, a and b are coefficients which determine the difference between the skeleton curve and the real curve.

Supposing the strain-stress curve of SMA is symmetrical about the point G (ε_0, σ_0), the strain-stress curve of SMA can be shown as follows:

$$\sigma - \sigma_0 = b_1(\varepsilon - \varepsilon_0) + b_2(\varepsilon - \varepsilon_0)^3 + b_3\left[1 - \left(\frac{\varepsilon - \varepsilon_0}{b_4}\right)^2\right]\dot{\varepsilon} \tag{2}$$

where σ is stress, ε is strain, b_i ($i = 1$, 2, 3, 4) are coefficients, skeleton curve is chosen as: $f_0(x) = b_1 x + b_2 x^3$. $b_4 = \varepsilon_0$ since the loading curve has the same value as the unloading curve when $\varepsilon = 0$, and $\sigma_0 - b_1\varepsilon_0 - b_2\varepsilon_0^3 = 0$ because the initial stress of SMA must be avoided for industry application, so Eq. 2 can be rewritten as follows:

$$\sigma = a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + (a_4\varepsilon - a_5\varepsilon^2)\dot{\varepsilon} \tag{3}$$

where: $a_1 = b_1 + 3b_2\varepsilon_0^2$, $a_2 = -3b_2\varepsilon_0$, $a_3 = b_2$, $a_4 = \frac{2b_3}{b_4}$, $a_5 = \frac{b_3}{b_4^2}$.

Model of SMA simply supported beam in axial stochastic excitation was shown in Fig. 2, where $w(x,t)$ is displacements of points of SMA beam, N is axial excitation, $N = N_0 - c_N\xi(t)$, N_0 is initial excitation, c_N is coefficient, $\xi(t)$ is Gauss white noise whose mean is zero and intensity is $2D$, $D > 0$.

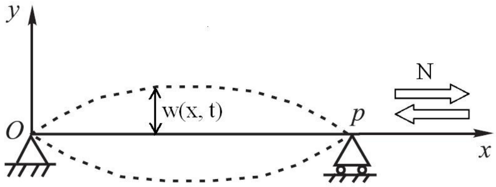

Fig. 2. The model of SMA simply supported beam in axial stochastic excitation

In this paper, tension and compression were assumed as symmetrical, so the neutral axis was located in the geometrical center. A rectangular cross-section under bending moment M was shown in Fig. 3.

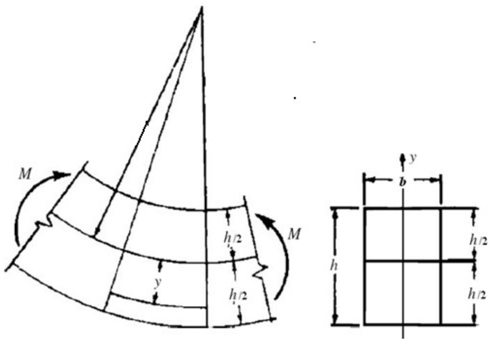

Fig. 3. The cross-section and curvature of the assumed beam

The boundary conditions of SMA simply supported beam can be written as follows:

$x = 0$: $w = 0$, $w_{xx} = 0$

$x = l$: $w = 0$, $w_{xx} = 0$

In main mode, w can be shown as follows:

$$w(t,x) = u(t)\sin\left(\frac{\pi}{l}x\right) \tag{4}$$

where $u(t)$ is amplitude of the fundamental mode.

The geometrical deformation condition is:

$$\varepsilon = -y\frac{\partial^2 w}{\partial x^2} \tag{5}$$

According to the relationship between stress and strain shown in Eq. 3, the bending moment M can be shown as follows:

$$M = \int -\sigma y dA = b\int_{-h/2}^{h/2} -y[a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + (a_4\varepsilon - a_5\varepsilon^2)\dot{\varepsilon}]dy \tag{6}$$

Considering Eq. 4 and Eq. 5, we obtained:

$$M = -I_1 a_1 \pi^2 u \sin\left(\frac{\pi}{l}x\right) - I_3 a_3 \pi^6 u^3 \sin^3\left(\frac{\pi}{l}x\right) + I_3 a_5 \pi^6 u^2 \sin^3\left(\frac{\pi}{l}x\right)\dot{u} \tag{7}$$

where $I_1 = \dfrac{bh^3}{12l^2}$, $I_3 = \dfrac{bh^5}{80l^6}$.

The dynamical equation of SMA simply supported beam can be shown as follows:

$$\frac{\partial^2 M}{\partial x^2} + N\frac{\partial^2 w}{\partial x^2} + c\frac{\partial w}{\partial t} + \rho A \frac{\partial^2 w}{\partial t^2} = 0 \tag{8}$$

where c is linear damping coefficient, ρ is density of the SMA and A is area of the cross-section of SMA beam.

Considering Eq. 4 and Eq. 7, we obtained differential equation of vibration amplitude with parametric excitation as follows:

$$\ddot{u} + \frac{a_1 I_1 \pi^4 - F_0 \pi^2}{\rho A l^2} u + \left(\frac{c}{\rho A} - \frac{3a_5 I_3 \pi^8}{4\rho A l^2}u^2\right)\dot{u} + \frac{3a_3 I_3 \pi^8}{4\rho A l^2}u^3 = \frac{F\pi^2}{\rho A l^2}u\xi(t) \tag{9}$$

Introducing non-dimensional transformation:

$$t^* = \left(\frac{a_1 I_1}{\rho A l^2}\right)^{\frac{1}{2}} t, \quad \Omega^* = \left(\frac{a_1 I_1}{\rho A l^2}\right)^{\frac{1}{2}} \Omega$$

and ignoring the symbol *, we obtained the non-dimensional motion equation as follows:

$$\ddot{u} + ku + \alpha u^3 + (\mu - \gamma u^2)\dot{u} = eu\xi(t) \tag{10}$$

where: $k = \pi^4 - \dfrac{N_0 \pi^2}{a_1 I_1}$, $p_0 = a_1 I_1 \pi^2$, $\alpha = \dfrac{3a_3 I_3 \pi^8}{4a_1 I_1}$, $\mu = \dfrac{cl}{\sqrt{\rho A a_1 I_1}}$, $\gamma = \dfrac{3a_5 I_3 \pi^8}{4l\sqrt{\rho A a_1 I_1}}$, $e = \dfrac{c_N \pi^2}{a_1 I_1}$.

Stochastic Stability Analysis of System

Let $u = q$, $\dot{u} = p$, Eq. 10 can also be shown as follows:

$$\begin{cases} \dot{q} = p \\ \dot{p} = -kq - \alpha q^3 - (\mu - \gamma q^2)p + eq\xi(t) \end{cases} \tag{11}$$

Considering that the items $-\alpha q^3$, $-(\mu - \gamma q^2)p$ and $eq\xi(t)$ are all small, the Hamiltonian function of Eq. 11 can be shown as follows:

$$H = \frac{1}{2}(p^2 + kq^2) \tag{12}$$

According to the quasi-nonintegrable Hamiltonian system theory, the Hamiltonian function $H(t)$ converges weakly in probability to an one-dimensional Ito diffusion process. The averaged Ito equation about the Hamiltonian function can be shown as follows:

$$dH = m(H)dt + \sigma(H)dB(t) \tag{13}$$

where $B(t)$ is standard Wiener process, $m(H)$ and $\sigma(H)$ are drift and diffusion coefficients of Ito stochastic process, which can be obtained in stochastic averaging method:

$$m(H) = \frac{De^2 - \mu k}{k}H - \frac{2a}{k^{3/2}\pi}H^2 + \frac{\gamma}{2k}H^2 \tag{14}$$

$$\sigma^2(H) = \frac{De^2}{k}H^2 \tag{15}$$

Based on quasi-nonintegrable Hamiltonian system theory [19], the largest Lyapunov exponent of a linearized system is defined as follows:

$$\lambda = \lim_{t\to\infty}\frac{1}{t}\ln\left\|Z(t, z_0)\right\| \tag{16}$$

The linearized Ito differential equation can be shown as follows after the system was linearized in the trivial solution $H = 0$:

$$dH = m'(0)Hdt + \sigma'(0)HdB(t) \tag{17}$$

Then the associated largest Lyapunov exponent is:

$$\lambda = \lim_{t\to\infty}\frac{1}{t}\ln H^{1/2} = \left\{m'(0) - \left[\sigma'(0)\right]^2/2\right\}/2 = \frac{De^2}{4k} - \frac{\mu}{2} \tag{18}$$

Now the local stochastic stability of the system can be discussed as follows:
1) The trivial solution $H = 0$ is locally asymptotic stable if and only if $\lambda < 0$, which means $\mu > \frac{De^2}{2k}$;
2) The trivial solution $H = 0$ is locally asymptotic unstable if and only if $\lambda > 0$, which means $\mu < \frac{De^2}{2k}$;
3) Bifurcation should appear near the trivial solution $H = 0$ if and only if $\lambda = 0$, which means $\mu = \frac{De^2}{2k}$.

The largest Lyapunov exponent can only estimate the local stability. In this paper, the boundary classification method was used to analyze the global stability of the trivial solution of the system. Generally, the boundaries of diffusion process are singular, and the boundary classification is often determined by diffusion exponent, drift exponent and character value [20].

When $H \to 0$:

$$m(H) = \frac{De^2 - \mu k}{k} H - \frac{2a}{k^{3/2}\pi} H^2 + \frac{\gamma}{2k} H^2 \to \frac{De^2 - \mu k}{k} H$$

$$\sigma^2(H) = \frac{De^2}{k} H^2 \to \frac{De^2}{k} H^2$$

So:

$$\alpha_l = 2\,,\ \beta_l = 1\,,\ c_l = 2\left(1 - \frac{\mu k}{De^2}\right)$$

where α_l is diffusion exponent, β_l is drift exponent, c_l is character value, l is left boundary.

Thus, the left boundary $H = 0$ belongs to the first kind of singular boundary. According to the classification for singular boundary [20], we obtained:

1) The left boundary $H = 0$ is repulsively natural if $c_l > 1$;

2) The left boundary $H = 0$ is strictly natural if $c_l = 1$;

3) The left boundary $H = 0$ is attractively natural if $c_l < 1$.

Similarly, the right boundary $H = \infty$ belongs to the second kind of singular boundary.

When $H \to \infty$:

$$m(H) = \frac{De^2 - \mu k}{k} H - \frac{2a}{k^{3/2}\pi} H^2 + \frac{\gamma}{2k} H^2 \to -\frac{2a}{k^{3/2}\pi} H^2 + \frac{\gamma}{2k} H^2$$

$$\sigma^2(H) = \frac{De^2}{k} H^2 \to \frac{De^2}{k} H^2$$

So:

$$\alpha_r = 2\,,\ \beta_r = 2\,,\ c_r = \frac{\sqrt{k}\pi\gamma - 4a}{2\sqrt{k}\pi De^2}$$

where r is the right boundary. Thus, the right boundary $H = \infty$ is an entrance boundary.

The necessary and sufficient conditions for globally asymptotic stability of the trivial solution require that the left boundary be attractively natural and the right boundary be entrance. Thus, the trivial solution $H = 0$ is globally asymptotically stable only $c_l < 1$, which means $\mu > \frac{De^2}{2k}$. The influence of the character value to the stability was shown in Fig. 4.

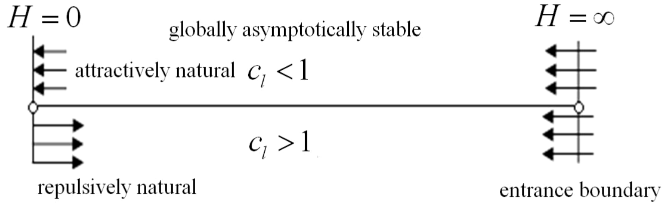

Fig. 4. The influence of the character value to the stability

Stochastic Bifurcation and Simulation

The averaged FPK equation of Eq. 11 is:

$$\frac{\partial f}{\partial t}=-\frac{\partial}{\partial H}[m(H)f]+\frac{1}{2}\frac{\partial^2[\sigma^2(H)f]}{\partial H^2} \tag{19}$$

where f is probability density.

Thus, the stationary probability density function of the system is:

$$f(H)=\overline{A}\exp\left(-\int_0^H\left\{\left[\frac{d\sigma^2}{dt}-2m(t)\right]/\sigma^2\right\}dt\right)=\overline{A}H^\eta\exp\left(\frac{2a}{k^{1/2}\pi De^2}H-\frac{\gamma}{2De^2}H\right) \tag{20}$$

where $\overline{A}$ is a normalization constant, $\eta=-2\frac{\mu k}{De^2}$.

The joint probability density function of the system is:

$$f(p,q)=A\left(\frac{1}{2}p^2+\frac{1}{2}kq^2\right)^\eta\exp\left[\frac{4a-\pi\sqrt{k}\gamma}{2\pi\sqrt{k}De^2}\left(\frac{1}{2}p^2+\frac{1}{2}kq^2\right)\right] \tag{21}$$

The results of numerical simulation were shown in Fig. 5 – Fig. 8, where $k=0.5$, $D=0.5$, $c=0.05$, $l=1$, $M=40$, $E=2\times10^{11}$, $A=8\times10^{-4}$, $I=6\times10^{-11}$.

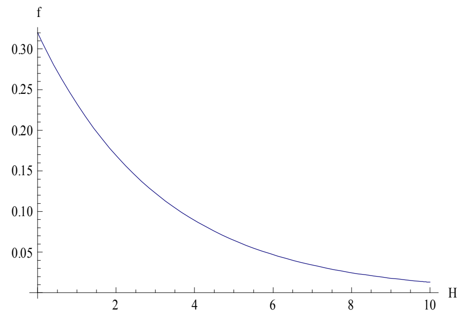

Fig. 5. The steady-state probability density when $\mu=0$

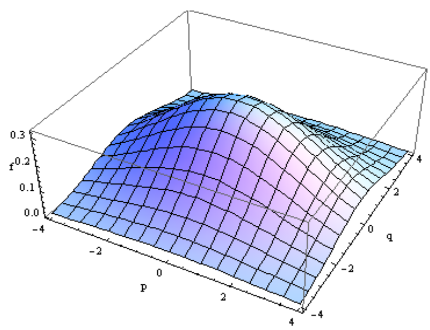

Fig. 6. The joint probability density when $\mu=0$

From Fig. 5 – Fig. 8, we can see that:

1) $p=0$ and $q=0$ when $H=0$ since $H=\frac{1}{2}\left(p^2+kq^2\right)$, so the trivial solution $H=0$ corresponds to the origin (0, 0) in the figure of joint probability density;

2) The steady-state probability density of $H = 0$ is the max when $\mu = 0$, which is locally asymptotic unstable;
3) The steady-state probability density of $H = 0$ decreases when μ increases, and its stability varies from unstable to stable;
4) Stochastic Hopf bifurcation appears when the bifurcation parameter μ varies. We can obviously see that there is a limit cycle in the figures of joint probability density, which is accord with the result of stochastic stability.

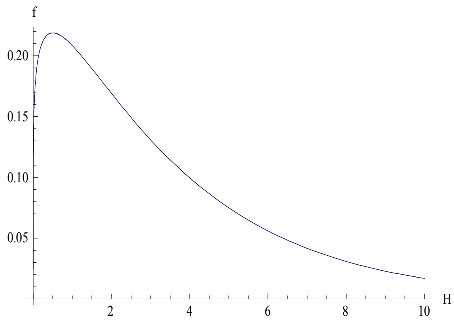

Fig. 7. The steady-state probability density when $\mu = 0.1$

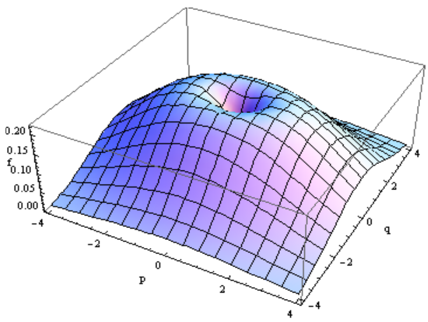

Fig. 8. The joint probability density when $\mu = 0.1$

Conclusions

In this paper, nonlinear dynamic characteristics of shape memory alloy (SMA) simply supported beam in axial stochastic excitation were studied. Von del Pol nonlinear difference item was introduced to interpret the hysteresis phenomenon of the strain-stress curve of SMA, and the hysteretic nonlinear dynamic model of SMA simply supported beam in axial stochastic excitation was developed. The local stochastic stability of the system was analyzed according to the largest Lyapunov exponent, and the global stochastic stability of the system was discussed in singular boundary theory. The steady-state probability density function and the joint probability density function of the system were obtained in quasi-nonintegrable Hamiltonian system theory. The result of simulation shows that the stability of the trivial solution varies from unstable to stable when the bifurcation parameter μ varies, and stochastic Hopf bifurcation appears in the process. The result is helpful to stochastic bifurcation control to SMA simply supported beam.

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