805. Propagation of weak waves in the inhomogeneous elastoviscoplastic medium with a cell structure

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(Received 5 April 2012; accepted 14 May 2012)

Abstract. Non-stationary acceleration waves in the fluid-saturated inhomogeneous elastoviscoplastic porous medium are studied using the mathematical theory of discontinuities. The equations for determining the intensity and the geometry of wave fronts of the fluid-saturated elastoviscoplastic medium were first derived. It is shown that in the medium under consideration there are two types of irrotational waves and one equivoluminal wave, that are equal to the velocities in the homogeneous elastic porous media at every point.

Keywords: biological tissues, elastoviscoplastic medium, wave velocities, equivoluminal waves, irrotational waves, porous medium.

List of Accepted Symbols

T_{ij} – full tension tensor of the porous medium;
m – porosity of the medium;
N – force acting on the fluid, and related to a unit of cross-section area of the porous medium;
L, μ – Lame coefficients;
k – yield stress of the material;
η – coefficient of viscosity;
$\vec{u}^{(1)}$ – displacement vector of an elastoviscoplastic phase (of the porous medium skeleton);
$\vec{u}^{(2)}$ – displacement vector of the fluid;
R_0 – compressibility modulus of the fluid;
ρ_{12} – coefficient of the dynamic connection of the elastoviscoplastic phase and the fluid;
ρ_1, ρ_2 – densities of the phases;
ρ_{11}, ρ_{22} – effective densities of the phases;
$u_i^{(\alpha)}$ – displacement components of the phases of the medium $(\alpha = 1,2)$;
$v_i^{(\alpha)}$ – velocity components of the phase displacements of the medium;
ν_i – components of the unit vector of the normal to wave surface $\sum(t)$;
Ω – average curvature of the wave surface;
K – Gaussian curvature of the wave surface;
$x_{j,\beta}$ – derivatives of Cartesian coordinates x_i by the curvilinear coordinates;
$g^{\alpha\beta}$ – coefficients of the first quadratic form;
$b_{\gamma\sigma}$ – coefficients of the second quadratic form;
$W = \sqrt{\omega\omega}$ – wave intensity.

Introduction

Biological tissues possess a well-marked cell structure, where filtration exchange processes take place through the membranes that restrain the cell. The cell membranes possess a layered structure and as a result the biological tissue in vivo can be considered as a porous medium saturated with filterable fluid. Phenomenologically such medium is described by the viscoelastic models of various types. Clearly expressed inhomogeneity and anisotropy of the biological tissues require the use of appropriate media models significantly complicating the analysis of wave propagation. Dynamic effect on the biological tissues is realized in the form of spreading or standing waves, which provide a double effect depending on their characteristics. High-intensity waves can damage, and low-intensity waves can provide stimulating, rehabilitating effects. In both cases irreversible effects are linked in isometric conditions and as a rule with irreversible deformations. Weak waves, which definition will be given hereafter, are used for biostimulation and rehabilitation. It is usually an ultrasound for well-balanced waves, and for nonstationary it is the so called waves of accelerations. Asymptotic methods such as the method of leaps, the ray method, the functionally invariant method, the method based on kinematic and dynamic Fermat's and Huygens' principles are the effective methods for the solution of wave equations for inhomogeneous media. According to these methods the displacement of the wave-packet can approximately be described as a displacement of the center of mass (energy transfer of the wave-packet) along the characteristic (of the ray), the equations of which can be found as the Lagrange–Euler equations from the Fermat's principle, and a set of centers of masses at any given time is defined as a surface (wavefront) according to the Huygens' principle. Thus, in general, it is possible to find the transfer equation of wave intensity along the selected ray for an inhomogeneous anisotropic medium that is important for the diagnostic of the effect on the biological tissues, allowing an accurate localization of the dynamic effects.

As it is known, the consideration of medium inhomogeneity is realized by using different models. The best known models are: 1 – when there is exactly or probabilistically one component at every point, 2 – when there are all the components at every point of the medium. The ray method is developed for the first model to solve different tasks of the wave propagation in solids the method for constructing solutions such as the discontinuity of field values at the front is being developed for the second model (this work).

The velocities of filtration processes, of diffusion in the biological tissues are much lower than the velocities of the wave propagation that is why in the first approximation we consider that medium inhomogeneity does not change within time. Waves may render irreversible changes for the medium that can be considered in the model of the medium with the help of models of plasticity. Thus, the consecutive effect of pulses, providing weak plastic deformation of the opposite sign, can bring the desired effect of reconstruction, when there is a wave rehabilitation of plastically deformed tissues. Macrostatically biological bodies have strongly marked anisotropy, but the medium can be considered as locally isotropic in the first approximation.

Wave propagation in complex media can be characterized by different processes: spatial and temporal propagation, nonlinear effects, which while interacting with each other provide sustainable or unsustainable energy propagation, distributed along the front. As it is known, the spatial propagation dilutes the front (packet) in width, the temporal propagation is connected with energy dissipation and leads to a decrease of the intensity (amplitude), and the nonlinearity counteracts these processes. We assume that there are stable fronts of spreading waves in the medium under consideration.

Under the assumptions made we managed for the first time to derive equations to describe the wave pattern of the propagation of the wave fronts in a fluid-saturated inhomogeneous elastoviscoplastic medium. It is stated that weak waves (acceleration waves) can exist as two types of equivoluminal waves and an irrotational one, and their velocities are equal to the velocities of the waves in homogeneous elastic porous media.

The propagation of elastic stationary and non-stationary waves in a homogeneous porous medium was considered in works [1-4]. In [5] acceleration waves in a fluid-saturated elastic porous medium are studied. In works [6, 7] the propagation and the attenuation of waves in an inhomogeneous viscous-elastoplastic medium without the saturation of fluid was investigated. Some aspects of the impact of aperiodic waves on the biological tissues are considered in [8], the wave propagation in randomly inhomogeneous media was investigated in [9].

Research Methodology

1. Let's consider the interpenetrating motion of an elastoviscoplastic and fluid phase as the motion of fluid in a deformable porous medium, the physical and mechanical characteristics of which are the functions of the coordinates. It is assumed that the pore sizes are small in comparison with the distance at which the kinematic and geometric characteristics of the motion change significantly. In this case, we can assume that the elastoviscoplastic and fluid phases are solid media and there will be two displacement vectors at every point of space: $\vec{u}^{(1)}-$ is the displacement vector of an elastoviscoplastic phase (of the porous medium skeleton) and $\vec{u}^{(2)}-$ is the displacement vector of the fluid.

The elastoviscoplastic phase is described by the Bingham's body model [6, 7]. In this case, we assume that the deformations of the phases are small and are represented for the elastoviscoplastic phase as a sum of elastic and plastic ones:

$$e_{ij}^{(1)} = e_{ij}^{(1)e} + e_{ij}^{(1)p}, \quad e_{ij}^{(1)e} = \frac{1}{2}(u_{i,j}^{(1)} + u_{j,i}^{(1)}), \quad e_{rr}^{(2)} = u_{r,r}^{(2)} \tag{1.1}$$

For the elastoviscoplastic porous medium the elastic deformation tensor is connected with the tension tensor generalized by Hooke's law [3, 5]:

$$T_{jk} = \lambda(x_i)e_{rr}^{(1)e}(x_i)\delta_{jk} + 2\mu(x_i)e_{jk}^{(1)e} + A_1(x_i)e_{rr}^{(2)} \tag{1.2}$$

$$N = A_1(x_i)e_{rr}^{(1)e} + A_2(x_i)e_{rr}^{(2)}$$

$$A_1 = a(x_i)R_0(x_i)\beta(x_i), \quad \lambda(x_i) = L(x_i) + \frac{\beta a^2}{m(x_i)}R_0(x_i)$$

$$A_2(x_i) = \beta m R_0, \quad a = 1 - m - \frac{K(x_i)}{K_0(x_i)}, \quad \beta = (1 + \frac{aR_0}{mK_0})^{-1}, \quad K = L + \frac{2}{3}\mu$$

where $N-$ is the force acting on the fluid, and related to a unit of cross-section area of the porous medium; L, $\mu-$ are Lame coefficients; $K-$ is the bulk modulus of a porous skeleton with empty pores; $m-$ is the porosity; $W_{1p} = W_{1p}^{(0)} + W_{1p}^{(1)} + W_{1p}^{(2)} + ...$ is the compressibility modulus of the fluid; K_0- is the true bulk modulus of the elastoviscoplastic phase; $\delta-$ is Kronecker symbol, index 1 at the top in parentheses refers to the elastoviscoplastic phase, index 2 refers to the fluid.

The tensor of plastic deformation velocity is connected with the tension tensor by a local condition of plasticity [6, 7]:

$$(S_{ij} - \eta\varepsilon_{ij}^{(1)p})(S_{ij} - \eta\varepsilon_{ij}^{(1)p}) = 2k^2, \quad S_{ij} = T_{ij} - \frac{1}{3}T_{kk}\delta_{ij} \tag{1.3}$$

and by the correlations of the associated flow law:

$$\varepsilon_{ij}^{(1)p} = \psi(S_{ij} - \eta\varepsilon_{ij}^{(1)p}) \tag{1.4}$$

$\psi > 0$ when $S_{ij}S_{ij} > 2k^2$, $\psi = 0$ when $S_{ij}S_{ij} \le 2k^2$

where $\eta(x_i)-$ is the coefficient of viscosity, $k(x_i)-$ is the yield stress of the material, $\psi(x_i)-$ is a positive factor.

Repeated Latin indices imply summation from one to three, according to the Greek - from one to two. The dot over the letter denotes the time derivative.

It follows from correlations (1.1) – (1.4) that:

$$\dot{T}_{jk} = \lambda v_{r,r}^{(1)}\delta_{jk} + \mu(v_{j,k}^{(1)} + v_{k,j}^{(1)}) - 2\mu\frac{\psi \cdot S_{jk}}{1+\eta\cdot\psi} + A_1 v_{r,r}^{(2)}\delta_{jk}$$
$$\dot{N} = A_1 v_{r,r}^{(1)} + A_2 v_{r,r}^{(2)}, \quad v_i^{(1)} = \dot{u}_i^{(1)}, \quad v_i^{(2)} = \dot{u}_i^{(2)} \tag{1.5}$$

Expressions (1.5) along with the equations of motion:

$$\rho_{11}\dot{v}_i^{(1)} + \rho_{12}\dot{v}_i^{(2)} = T_{ik,k}, \quad \rho_{12}\dot{v}_i^{(1)} + \rho_{22}\dot{v}_i^{(2)} = N_{,i}$$
$$\rho_{11} = \rho_1 - \rho_{12}, \quad \rho_{22} = \rho_2 - \rho_{12} \tag{1.6}$$

where $\rho_{12}(x_i)-$ is the coefficient of the dynamic connection of the elastoviscoplastic phase and the fluid, $\rho_1(x_i)$ and $\rho_2(x_i)-$ are the densities of the phases, $\rho_{11}(x_i)$ and $\rho_{22}(x_i)-$ are the effective densities of the phases, $v_i^{(\alpha)}$, $(\alpha = 1,2)-$ are the displacement velocities of the phases.

The acceleration wave in the considered porous medium is determined by isolated surface $\sum(t)$, on which the voltage, the force acting on the fluid and the displacement velocities of the phases are continuous, and their particular derivatives undergo discontinuity. The physical and mechanical parameters of the medium and their gradients are continuous.

We apply the mathematical theory of discontinuities to correlations (1.5) and (1.6) and considering the geometric and kinematic compatibility conditions of first order for the phases [8], we obtain the system of equations:

$$(\lambda+\mu)\lambda_j^{(1)}\nu_i\nu_j + \mu\lambda_i^{(1)} + A_1\lambda_j^{(2)}\nu_i\nu_j = \rho_{11}G^2\lambda_i^{(1)} + \rho_{12}G^2\lambda_i^{(2)} \tag{1.7}$$

$$A_1\lambda_j^{(1)}\nu_i\nu_j + A_2\lambda_j^{(2)}\nu_i\nu_j = \rho_{12}G^2\lambda_i^{(1)} + \rho_{22}G^2\lambda_i^{(2)}$$

where $\lambda_i^{(\alpha)}$ $(\alpha = 1;2)-$ is the values characterizing the leaps of the first derivatives of the displacement velocities of the phases; $G-$ is the velocity of the wave surface; $\nu_i -$ is the components of the unit normal of the wave surface.

Let us assume that on the wave surface $\lambda_i^{(2)}\nu_i = \omega_2 \neq 0$, let's multiply each equation of system (1.7) by ν_i and sum over the index. As a result, we obtain a homogeneous system of equations relative to ω_1 and ω_2:

$$(\Lambda_\alpha - \rho_{1\alpha}G^2)\omega_1 + (A_\alpha - \rho_{\alpha 2}G^2)\omega_2 = 0, \ \ (\alpha = 1,2) \qquad (1.8)$$
$$\Lambda_1 = \lambda + 2\mu, \ \ \Lambda_2 = A_1$$

The condition for the existence of non-zero solutions of the system (1.8) is that its determinant must be zero. This condition leads (1.8) to the equation relative to the velocity of irrotational waves $(\lambda_i^{(\alpha)}\nu_i = \omega_\alpha \neq 0, \ \alpha = 1;2, \ G = G_l \ )$:

$$(\rho_{11}\rho_{22} - \rho_{12}^2)G_l^4 + (2\rho_{12}A_1 - \rho_{11}A_2 - \rho_{22}\Lambda_1)G_l^2 + A_2\Lambda_1 - A_1^2 = 0 \qquad (1.9)$$

Equation (1.9) implies that irrotational waves of two types G_{l_1} and G_{l_2} propagate in the fluid-saturated elastoviscoplastic inhomogeneous porous medium, and the velocity squared of these waves is given by the formula:

$$G_l^2 = \frac{1}{2k_5}(k_1 \pm \sqrt{k_2^2 - 4k_3k_4}), \ \ (\lambda_i^{(\alpha)}\nu_i \neq 0), \ k_1 = \rho_{11}A_2 - 2\rho_{12}A_1 + \rho_{22}\Lambda_1,$$
$$k_2 = \rho_{11}A_2 - \rho_{22}\Lambda_1, \ \ k_3 = \rho_{22}A_1 - \rho_{12}A_2, \ \ k_4 = \rho_{12}\Lambda_1 - \rho_{11}A_1, \ \ k_5 = \rho_{11}\rho_{22} - \rho_{12}^2 \qquad (1.10)$$

If $\lambda_i^{(\alpha)}\nu_i = 0 \ \ (\alpha = 1;2)$, on the wave surface, on the condition that not all $\lambda_i^{(\alpha)}$ are equal to zero simultaneously, then we obtain the formula for determining the velocity of the equivoluminal wave $(G = G_t)$ from the system (1.7):

$$G_t = \sqrt{\frac{\mu\rho_{22}}{\rho_{11}\rho_{22} - \rho_{12}^2}} \ \ (\lambda_i^{(\alpha)}\nu_i = 0) \qquad (1.11)$$

Thus, in this inhomogeneous elastoviscoplastic porous medium there are two types of irrotational waves and one equivoluminal, which possess the velocities of longitudinal and transversal waves at every point of the medium [4].

2. Let's get the equations that determine the intensity change of equivoluminal and irrotational waves. For this equation of motion (1.6), taking into account correlations (1.5) and the second equality (1.3), let's write in the discontinuities:

$$\lambda[v_{j,ij}^{(1)}] + \mu([v_{i,jj}^{(1)}] + [v_{j,ij}^{(1)}]) + \lambda_{,i}[v_{j,j}^{(1)}] + \mu_{,j}([v_{i,j}^{(1)}] + [v_{j,i}^{(1)}]) - \frac{2\mu\psi}{1+\eta\psi}([T_{ij,j}] - \frac{1}{3}[T_{kk,i}]) -$$
$$- \frac{2\mu S_{ij}}{(1+\eta\psi)^2}[\frac{\partial\psi}{\partial x_j}] + A_1[v_{j,ij}^{(2)}] + A_{1,i}[v_{j,j}^{(2)}] = \rho_{11}[\ddot{v}_i^{(1)}] + \rho_{12}[\ddot{v}_i^{(2)}] \qquad (2.1)$$

$$A_1[v_{j,ij}^{(1)}] + A_2[v_{j,ij}^{(2)}] + A_{1,i}[v_{j,j}^{(1)}] + A_{2,i}[v_{j,j}^{(2)}] = \rho_{12}[\ddot{v}_i^{(1)}] + \rho_{22}[\ddot{v}_i^{(2)}]$$

Let's write the geometric and kinematic compatibility conditions of the second order for the phases [8]:

$$[\ddot{v}_i^{(\alpha)}] = L_i^{(\alpha)} G_l^2 - 2G_l \frac{\delta\lambda_i^{(\alpha)}}{\delta t} - \lambda_i^{(\alpha)} \frac{\delta G_l}{\delta t}$$

$$[v_{j,ij}^{(\alpha)}] = L_j^{(\alpha)} \nu_i \nu_j + g^{\gamma\beta} \lambda_{j,\gamma}^{(\alpha)} (x_{j,\beta}\nu_i + x_{i,\beta}\nu_j) - \lambda_j^{(\alpha)} g^{\gamma\beta} g^{\sigma\tau} b_{\gamma\sigma} x_{i,\beta} x_{j,\tau} \tag{2.2}$$

$$[v_{i,jj}^{(\alpha)}] = L_i^{(\alpha)} - 2\Omega_l \lambda_i^{(\alpha)}, \ [v_{i,j}^{(\alpha)}] = \lambda_i^{(\alpha)} \nu_j, \ [T_{ij,j}] = \mu_{ij}\nu_j, \ \alpha = 1, \ 2, \ l = l_1, l_2$$

Here μ_{ij}, $L_i^{(\alpha)}-$ are respectively the values characterizing the leaps of the first derivatives of the tensions and the second derivatives of the displacement velocities of the phases, $\Omega_l -$ is the average curvature of the wave surface of the irrotational wave, $g^{\gamma\beta}, b_{\gamma\sigma} -$ are the coefficients of the first and second quadratic forms, $x_{i,\beta} -$ are the derivatives of Cartesian coordinates x_i by the curvilinear coordinates u_β of the wave surface, $\delta -$ differentiation with respect to time t. Value μ_{ij} can be found from the first equation (1.6).

Let's substitute expressions (2.2) into equality (2.1), multiply by ν_i, and taking into account that $\nu_i\nu_i = 1$, $x_{i,\beta}\nu_i = 0$, we have:

$$L_i^{(1)} \nu_i B_{1\alpha} - 2\rho_{1\alpha} G_l \frac{\delta\omega_1}{\delta t} - \rho_{1\alpha} \frac{\delta G_l}{\delta t}\omega_1 + (2\Lambda_\alpha \Omega_l - \Lambda_{\alpha,i}\nu_i)\omega_1 +$$

$$+ L_i^{(2)} \nu_i B_{\alpha 2} - 2\rho_{2\alpha} G_l \frac{\delta\omega_2}{\delta t} - \rho_{2\alpha} \frac{\delta G_l}{\delta t}\omega_2 + (2A_\alpha \Omega_l - A_{\alpha,i}\nu_i)\omega_2 + \tag{2.3}$$

$$+ \left\{ \frac{2\mu S_{ij}\nu_i\nu_j}{(1+\eta\psi)^2} \left[\frac{d\psi}{dn}\right] - \frac{4}{3}\frac{\mu\psi G_l^2}{1+\eta\psi}(\rho_{11}\omega_1 + \rho_{12}\omega_2) \right\} (2-\alpha) = 0$$

$$B_{1\alpha} = \rho_{1\alpha} G_l^2 - \Lambda_\alpha, \ B_{1\alpha} = \rho_{1\alpha} G_l^2 - \Lambda_\alpha, \ \alpha = 1, \ 2$$

Let's eliminate value $L_i^{(2)}$ from the system (2.3) in a standard way when $\alpha = 1, \ 2$. Then value $L_i^{(1)}$ will be eliminated from (2.3) taking into account equation (1.9).

After the transformations, the system of equations (2.3) is reduced to one equation with two unknown quantities ω_1 and ω_2:

$$2G_l(-k_5 G_l^2 + c_1)\frac{\delta\omega_1}{\delta t} + \{(-\rho_{11}\frac{\delta G_l}{\delta t} + 2\Lambda_1\Omega_l - \Lambda_{1,i}\nu_i)B_{22} -$$

$$-(-\rho_{12}\frac{\delta G_l}{\delta t} + 2A_1\Omega_l - A_{1,i}\nu_i)B_{12} - \frac{4}{3}\cdot\frac{\mu\psi G_l^2}{1+\eta\psi}\rho_{11}B_{22}\}\omega_1 + \tag{2.4}$$

$$+2G_l c_2 \frac{\delta\omega_2}{\delta t} + \{(-\rho_{12}\frac{\delta G_l}{\delta t} + 2A_1\Omega_l - A_{1,i}\nu_i)B_{22} -$$

$$-(-\rho_{22}\frac{\delta G_l}{\delta t} + 2A_2\Omega_l - A_{2,i}\nu_i)B_{12} - \frac{4}{3}\cdot\frac{\mu\psi G_l^2}{1+\eta\psi}\rho_{12}B_{22}\}\omega_2 +$$

$$+\frac{2\mu S_{ij}\nu_i\nu_j}{(1+\eta\psi)^2}B_{22}[\frac{d\psi}{dn}] = 0$$

$$k_5 = \rho_{11}\rho_{22} - \rho_{12}^2,\ \ c_1 = \rho_{11}A_2 - \rho_{12}A_1,\ \ c_2 = \rho_{12}A_2 - \rho_{22}A_1$$

With the help of equality (1.8) when $\alpha = 1$ we eliminate the value $\omega_2 = -\frac{B_{11}}{B_{12}}\omega_1 = \Gamma_l\omega_1$ from the equation (2.4). Then, after the transformations we get the differential equation for the change of the intensity of irrotational waves $W_{1l} = \sqrt{\omega_1\omega_1}$ in the inhomogeneous elastoviscoplastic porous medium of the first phase:

$$\frac{dW_{1l}}{ds} + \left\{\frac{\beta_2}{2G_l}\cdot\frac{dG_l}{ds} - \Omega_l + \frac{D_{,i}\nu_i}{2G_l^2 D_1} - \frac{2}{3}\cdot\frac{\mu\psi}{1+\eta\psi}\cdot\frac{D_2}{D_1}\right\}W_{1l} -$$

$$-\frac{\mu S_{ij}\nu_i\nu_j}{(1+\eta\psi)^2}\cdot\frac{B_{12}B_{22}}{G_l^2 D_1}\cdot[\frac{d\psi}{dn}] = 0 \quad (2.5)$$

$$D_1 = (\rho_{11}\rho_{12}A_2 + \rho_{12}\rho_{22}\Lambda_1 - 2\rho_{11}\rho_{22}A_1)G_l^2 + \rho_{11}A_1A_2 - 2\rho_{12}A_2\Lambda_1 + \rho_{22}A_1\Lambda_1$$
$$D_2 = (\rho_{11}\rho_{22}A_1 - \rho_{12}\rho_{22}\Lambda_1)G_l^2 + \rho_{12}A_2(\Lambda_1 - A_1)$$
$$D_{,i} = (\rho_{11}\rho_{12}A_{2,i} - 2\rho_{11}\rho_{22}A_{1,i} + \rho_{12}\rho_{22}\Lambda_{1,i})G_l^4 + (2\rho_{11}A_2A_{1,i} + 2\rho_{22}\Lambda_1A_{1,i} -$$
$$-\rho_{12}A_2\Lambda_{1,i} - \rho_{22}A_1\Lambda_{1,i} - \rho_{12}\Lambda_1A_{2,i} - \rho_{11}A_1A_{2,i})G_l^2 + A_1A_2\Lambda_{1,i} - 2A_2\Lambda_1A_{1,i} + A_1\Lambda_1A_{2,i}$$
$$\beta_1 = (\rho_{22}A_1 - \rho_{12}A_2)(\rho_{12}\Lambda_1 - \rho_{11}A_1),\ \ \beta_2 = -\{A_1D_1 + (4\beta_1 - \rho_{12}D_1)G_l^2\}(D_1B_{12})^{-1}$$

In (2.5) it is taken into account that $\frac{\delta}{\delta t} = G_l\frac{d}{ds}$, where $s-$ is the distance along the normals to the surface $\sum(t_0)$.

Let's find the value $[\frac{d\psi}{dn}]$ from the condition of plasticity (1.3):

$$[\frac{d\psi}{dn}] = -\frac{(\rho_{11}\omega_1 + \rho_{12}\omega_2)G_lS_{ij}\nu_i\nu_j}{3k^2\eta(1+\eta\psi)} \quad (2.6)$$

Then (2.5) can be written down as:

$$\frac{dW_{1l}}{ds}=\left\{\Omega_l-\frac{\beta_2}{2G_l}\cdot\frac{dG_l}{ds}-\frac{1}{2\gamma_1}\cdot\frac{dD}{ds}+\frac{\mu}{3(1+\eta\psi)D_1}\left(\frac{S_{ij}S_{ij}a_1}{(1+\eta\psi)k^2\eta G_l}+2\psi D_2\right)\right\}W_{1l}=0$$
$$a_1=B_{22}(\rho_{11}A_1-\rho_{12}\Lambda_1),\ \gamma_1=G_l^2D_1 \tag{2.7}$$

We find the intensity of irrotational waves in the elastoviscoplastic porous medium of the second phase from the expression:

$$W_{2l}=\Gamma_l W_{1l},\ W_{2l}=\Gamma_l W_{1l} \tag{2.8}$$

The overall intensity of irrotational waves in the fluid-saturated inhomogeneous elastoviscoplastic porous medium will be as the sum:

$$W_l=W_{1l}+W_{2l}=(1+\Gamma_l)W_{1l},\ (l=l_1,\ l_2) \tag{2.9}$$

To determine the intensity of an equivoluminal wave, let's differentiate with respect to variable α correlations $\lambda_j^{(1)}v_j=0,\ \lambda_j^{(2)}v_j=0$ that are running on the surface of the wave, and we obtain:

$$\lambda_{j,\alpha}^{(1)}v_j=-\lambda_j^{(1)}v_{j,\alpha}=\lambda_j^{(1)}g^{\sigma\tau}b_{\sigma\alpha}x_{j,\tau},\ \lambda_{j,\alpha}^{(2)}v_j=-\lambda_j^{(2)}v_{j,\alpha}=\lambda_j^{(2)}g^{\sigma\tau}b_{\sigma\alpha}x_{j,\tau} \tag{2.10}$$

Then let's write down the geometric compatibility conditions of the second order for the phases in the form:

$$[v_{j,ij}^{(1)}]=L_j^{(1)}v_iv_j+\lambda_{j,\alpha}^{(1)}g^{\alpha\beta}x_{j,\beta}v_i,\ [v_{j,ij}^{(2)}]=L_j^{(2)}v_iv_j+\lambda_{j,\alpha}^{(2)}g^{\alpha\beta}x_{j,\beta}v_i \tag{2.11}$$

We'll get a differential equation that determines the change of the intensity of the equivoluminal wave $(G=G_t)$ in the first phase in the process of its propagation, having substituted the geometric and kinematic compatibility conditions (2.11) in equalities (2.1), and taking into account (1.11):

$$\frac{dW_{1t}}{ds}=\left\{\Omega_t-\frac{1}{2}\cdot\frac{d\ln(\mu G_t)}{ds}-\frac{\rho_{11}G_t}{2(+\eta\psi)}\cdot\left(\frac{S_{ij}S_{ij}}{k^2\eta(1+\eta\psi)^2}+2\psi\right)\right\}W_{1t}$$
$$W_{1t}=\sqrt{\lambda_i^{(1)}\lambda_i^{(1)}} \tag{2.12}$$

where Ω_l – is the average curvature of the wave surface of the irrotational wave.

Changes of the intensity of the equivoluminal wave in the second phase can be found from the second equality (1.7):

$$W_{2t}=\Gamma_tW_{1t},\ \Gamma_t=-\frac{\rho_{12}}{\rho_{22}} \tag{2.13}$$

Then the change of the intensity of the equivoluminal wave in the fluid-saturated inhomogeneous elastoviscoplastic porous medium will be written down as:

$$W_t = W_{1t} + W_{2t} = (1+\Gamma_t)W_{1t} \tag{2.14}$$

Equations (2.7) and (2.12) can be written in one form:

$$\frac{dW_{1p}}{ds} = \left\{\Omega_p(s) - \frac{1}{2}(\chi_p(s) + g_p(s) - 2\xi_p(s))\right\}W_{1p}(s), \ \ (p = l,\ t) \tag{2.15}$$

$$\chi_l = \beta_2 \frac{dG_l}{ds}, \ \ \chi_t = \frac{d\ln G_t}{ds}, \ \ g_l = \frac{1}{\gamma_1}\cdot\frac{dD}{ds}, \ \ g_t = \frac{d\ln\mu}{ds}$$

$$\xi_l = \frac{\mu}{3(1+\eta\psi)D_1}\left(\frac{S_{ij}S_{ij}}{k^2\eta(1+\eta\psi)^2} + 2\psi D_2\right),$$

$$\xi_t = -\frac{\rho_{11}G_t}{2(1+\eta\psi)}\left(\frac{S_{ij}S_{ij}}{k^2\eta(1+\eta\psi)^2} + 2\psi\right)$$

As an unknown function equations (2.15) contain geometric invariant $\Omega_p(s) -$ the average curvature of wave surface $\sum(t)$, that changes in the process of the wave propagation, and consequently they are not closed. The equation for determining the average curvature and the method of solution is given in [5].

Since the wave fronts propagate along the rays, remaining all the time orthogonal to these rays, then the equation of the ray trajectory is found from the principle of Fermat functional [9,10]:

$$\frac{d\nu_i}{ds} = -g^{\alpha\beta}(\ln G_p)_{,\alpha}\, x_{i,\beta}, \ \ \frac{dx_{i,\beta}}{ds} = (\ln G_p)_{,\alpha}\nu_i - g^{\delta\gamma}b_{\delta\alpha}x_{i,\gamma}, \ \ \nu_i = \frac{dx_i}{ds} \tag{2.16}$$

where $x_{i,\beta} -$ is the vector tangent to the wave surface

3. Let's determine the intensity of the waves W_p $(p = l,\ t)$. For this, let's consider equation (2.15) and initial conditions:

$$W_{1p}^{(0)}(0) = W_{01p}^{(0)}, \ \ W_{1p}^{(i)}(0) = 0, \ \ i = 1, 2 \tag{3.1}$$

The solution of equation (2.15) we will find by the method of successive approximations.

Let's substitute the expressions:

$$W_{1p} = W_{1p}^{(0)} + W_{1p}^{(1)} + W_{1p}^{(2)} + \ldots.., \ \Omega_p = \Omega_p^{(0)} + \Omega_p^{(1)} + \Omega_p^{(2)} + \ldots.$$

in (2.15) and restricting to the second approximation, we'll get:

$$\frac{dW_{1p}^{(0)}}{ds} = \Omega_p^{(0)}W_{1p}^{(0)}, \ \ \frac{dW_{1p}^{(1)}}{ds} = \Omega_p^{(0)}W_{1p}^{(1)} + \left\{\Omega_p^{(1)} - \frac{1}{2}(\chi_p^{(1)} + g_p^{(1)} - 2\xi_p^{(1)})\right\}W_{1p}^{(0)} \tag{3.2}$$

$$\frac{dW_{1p}^{(2)}}{ds} = \Omega_p^{(0)} W_{1p}^{(2)} + \left\{ \Omega_p^{(1)} - \frac{1}{2}(\chi_p^{(1)} + g_p^{(1)} - 2\xi_p^{(1)}) \right\} W_{1p}^{(1)} + \Omega_p^{(2)} W_{1p}^{(0)}$$

where $\chi_p^{(1)}$, $g_p^{(1)}$, $\xi_P^{(1)}$ – denote the first orders of approximation of functions χ_p, g_p, ξ_P.

Zero approximation for Gaussian curvature corresponds to the homogeneous elastic porous medium [4]:

$$\Omega_p^{(0)} = \frac{\Omega_{0p} - K_{0p} s}{1 - 2\Omega_{0p} s + K_{0p} s^2} \tag{3.3}$$

where Ω_{0p} and K_{0p} – are the average and Gaussian curvatures of wave surface from which the distance s_0 is measured.

We'll write down the solution of equations (3.2) with initial conditions (3.1) and $\Omega_p^{(1)}(0) = 0$ as:

$$W_{1p}^{(0)} = \frac{W_{01p}^{(0)}}{\sqrt{1 - 2\Omega_{0p} s + K_{0p} s^2}}, \quad W_{1p}^{(1)} = \frac{W_{01p}^{(0)}}{\sqrt{1 - 2\Omega_{0p} s + K_{0p} s^2}} \int_0^s f^{(1)}(s_1) ds_1$$

$$W_{1p}^{(2)} = \frac{W_{01p}^{(0)}}{\sqrt{1 - 2\Omega_{0p} s + K_{0p} s^2}} \left\{ \int_0^s \Omega_p^{(2)}(s_2) ds_2 + \frac{1}{4} \int_0^s \int_0^s f^{(1)}(s_1) f^{(1)}(s_2) ds_1 ds_2 \right\} \tag{3.4}$$

$$f^{(1)}(s) = 2\xi_p^{(1)}(s) - (\chi_p^{(1)}(s) + g_p^{(1)}(s))$$

Then the intensity of two irrotational waves as well as of the equivoluminal waves in the fluid-saturated inhomogeneous elastoviscoplastic porous medium is determined by the formula:

$$W_p = W_{1p} + W_{2p}, \quad p = l, t \tag{3.5}$$

4. Example. Let's consider the fluid-saturated inhomogeneous elastoviscoplastic porous medium, characterized by elastic moduli $L(x)$, $\mu(x)$, the coefficient of the dynamic connection of the elastoviscoplastic phase and the fluid $\rho_{12}(x)$, the effective densities of the elastoviscoplastic phase $\rho_{12}(x)$ and the fluid $\rho_{22}(x)$, the viscosity $\eta(x)$ and the yield stress of the material $k(x)$.

The front of the irrotational wave with velocity (1.10) propagates at the time moment $t = 0$ in the x, y plane along the x axis. Determine the intensity of the given wave.

Since in this case, the quadratic forms are $g^{\alpha\beta} = b_{\alpha\beta} = 0$ and the curvatures are $\Omega_{0l} = K_{0l} = 0$ when $x = 0$, it follows that $\Omega_l = K_l = 0$. Then from (2.7) and (2.9) we find the dependence of the intensity level of the wave on the velocity and the physical-mechanical characteristics of the fluid-saturated inhomogeneous elastoviscoplastic porous medium:

$$W_l = (1 + \Gamma_l(x)) W_{0l} \exp\{-\frac{1}{2} \int_0^x [\beta_2(x) \frac{d \ln G_l(x)}{dx} + \frac{1}{\gamma_1(x)} \frac{dD(x)}{dx} - \tag{4.1}$$

$$-2\left(\frac{\mu(x)}{3n(x)D_1(x)}\left(\frac{s(x)s(x)}{n^2(x)k^2(x)\eta(x)}+\psi(x)D_2(x)\right)\right]dx\right\},\quad n(x)=1+\eta(x)\psi(x)$$

where $W_{0l}-$ is the function value W_l when $x=0$.

Specifying a particular type of inhomogeneity for the physical and mechanical characteristics of the medium, we find the change of the intensity level of the irrotational wave in the fluid-saturated inhomogeneous elastoviscoplastic porous medium.

Conclusions

1. Based on the principles of Fermat and Huygens propagation process of the wave packet in a heterogeneous viscoelastoplastic environment regarded as transfer of energy along the beam path in the form of the jump intensity, localized at the wave front.
2. On the kinematic and geometrical conditions compatibility, viscoelastoplastic environment is saturated with fluid, there are irrotational and equivoluminal waves.
3. Expressions for the wave velocities, depending on the environment parameters are provided.
4. A system of differential equations for wave energy transfer along the path of rays is provided, changing the geometric parameters of the energy flow lines and geometry of the wave front.

Acknowledgments

The presented work was funded by the Grant (TAP LB 18/2012) from Research Council of Lithuania and the Grant (T11ЛИТ-030) from the Foundation for Fundamental Research of the Republic of Belarus (ГКНТ-Литва-2011).

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