747. Analytical approximation of nonlinear frequency of cantilever beam vibrations

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Abstract. This research presents the application of modern analytical approaches for the nonlinear vibrations of cantilever beams. These methods are Homotopy Analysis Method, Parameter Expansion Method and Bubnov-Galerkin Weighted Residual Method. Powerful analytical methods are used to obtain frequency-amplitude relationship of dynamic behavior of the mentioned system. It is demonstrated that one term in series expansions of all methods are sufficient to obtain a highly accurate solution. Finally, a comparison with numerical methods is provided in order to confirm the soundness of the obtained results.

Keywords: homotopy analysis method, He's parameter expanding method, Bubnov-Galerkin method, nonlinear vibration of beam.

Introduction

Common structures such as aircraft wings, bridges, buildings etc. can be modeled as beams and their significant applications, makes necessary the study their dynamic behavior at large amplitudes. Many investigators have studied nonlinear vibrations of beams [1-22]. These research works predict the nonlinear frequencies of the beams, which are very important for the design of many engineering structures. An exact formulation of the beam problem was first investigated in terms of general elasticity equations [23]. The problem of the transversely vibrating beam was formulated in terms of the partial differential equation of motion, an external forcing function by many researchers.

In recent times, substantial progresses had been made in analytical solutions for nonlinear equations without small parameters [24-40]. There have been several classical approaches employed to solve the governing nonlinear differential equations to study the nonlinear vibrations including perturbation methods, form function approximations, semi-analytical finite element, artificial small parameter, energy balance method, Adomian's decomposition, variational iteration method, frequency amplitude formulation, HAM, multiple scales method, homotopy perturbation method (HPM) and He's parameter expanding method. The application of new equivalent function for dead-zone nonlinearity on the dynamical behavior of beam vibration using HPEM has been investigated by [15].

In this work the nonlinear ordinary differential equation of beam vibration is extracted from partial differential equation with first mode approximation, based on a Galerkin theory. The results presented in this paper exhibit that these analytical methods are very effective and convenient for nonlinear beam vibration for which the highly nonlinear governing equations exist. The proposed analytical methods demonstrate that one term in series expansions is sufficient to obtain a highly accurate solution of beam vibration.

Equation of motion

Let us consider a beam shown in Fig. 1 with the span L, mass per unit length of the beam m, moment of inertia of the cross-section I, modulus of elasticity E and let us assume that the Euler-Bernoulli theorem can be accepted. The flexural in-plane vibration of the cantilever beam is, as follows [18]:

$$m\ddot{v}+EIv^{iv}+\frac{1}{2}m\left\{v'\int_L^x\left[\frac{\partial^2}{\partial t^2}\int_0^x v'^2dx\right]dx\right\}'+EI\left[v'\left(v'v''\right)'\right]'=0 \tag{1}$$

Here, x is the axial coordinate measured from the origin, v denotes the lateral vibration in y direction. Assuming $v(x,t)=q(t)\varphi(x)$, where $\varphi(x)$ is the first eigenmode of the clamped-free beam and can be expressed as:

$$\varphi(x)=\cosh(\lambda x)-\cos(\lambda x)-\frac{\cosh(\lambda L)+\cos(\lambda L)}{\sinh(\lambda L)+\sin(\lambda L)}\left(\sinh(\lambda x)-\sin(\lambda x)\right) \tag{2}$$

where $\lambda=1.875$ is the root of characteristic equation for the first eigenmode. Applying the weighted residual Bubnov-Galerkin method yields:

$$\int_0^L\left[m\ddot{v}+EIv^{iv}+EI\left[v'\left(v'v''\right)'\right]'+\frac{1}{2}m\left\{v'\int_L^x\left[\frac{\partial^2}{\partial t^2}\int_0^x v'^2dx\right]dx\right\}'\varphi(x)\right]dx=0 \tag{3}$$

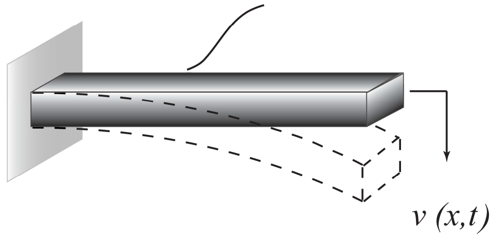

Fig. 1. Cantilever beam model

So, we can obtain the nonlinear equation in terms of the time-dependent variables as:

$$\ddot{q}+\beta_1 q+\beta_2 q^3+\beta_3 q\dot{q}^2+\beta_4 q^2\ddot{q}=0 \tag{4}$$

where:

$$\beta_1=12.3624EI/mL^4,\ \beta_2=40.44\,EI/mL^6,\beta_3=\beta_4=4.6/L^2 \tag{5}$$

Four analytical methods are employed to solve nonlinear ordinary Eq. (4) analytically.

Overview of the analytical procedures

1. He's Parameter Expanding Method

Consider the Eq. (4) for the vibration of a cantilever Euler-Bernoulli beam in the following form:

$$1.\ddot{q}+\beta_1 q+1.\left[\beta_2 q^3+\beta_3 q\dot{q}^2+\beta_4 q^2\ddot{q}\right]=0 \tag{6}$$

with the following general initial conditions:

$$q(0) = A\,,\quad \dot{q}(0) = 0 \tag{7}$$

The limit-cycles of oscillating systems are periodic motions with the period $T = 2\pi/\omega$, and thus $q(t)$ can be expressed by such a set of base functions:

$$\cos(m\omega t),\ m = 1, 2, 3, ... \tag{8}$$

We denote the angular frequency of oscillation by ω and note that one of our major tasks is to determine $\omega(A)$, that is, the functional behavior of ω as a function of the initial amplitude A. In the HPEM, an artificial perturbation equation is constructed by embedding an artificial parameter $p \in [0,1]$ which is used as an expanding parameter.

According to HPEM, the solution of Eq. (6) is expanded into a series of p in the form:

$$q(t) = q_0(t) + pq_1(t) + p^2 q_2(t) + ... \tag{9}$$

The coefficients *1* and β_1 in Eq. (4) are expanded in a similar way:

$$\begin{aligned} 1 &= 1 + pa_1 + p^2 a_2 + ... \\ \beta_1 &= \omega^2 - pb_1 - p^2 b_2 + ... \\ 1 &= pc_1 + p^2 c_2 + ... \end{aligned} \tag{10}$$

where $a_i,\ b_i,\ c_i\ (i = 1, 2, 3, ...)$ are to be determined. When $p = 0$, Eq. (6) becomes a linear differential equation, for which an exact solution can be calculated for $p = 1$. Substituting Eqs. (10) and (9) into Eq. (6), we have:

$$\begin{aligned} &(1+pa_1)(\ddot{q}_0 + p\ddot{q}_1) + (\omega^2 - pb_1)(q_0 + pq_1) + (pc_1 + p^2 c_2)\Big[\beta_2 (q_0 + pq_1)^3 + \beta_3 (q_0 + pq_1)(\dot{q}_0 + p\dot{q}_1)^2 + ... \\ &... + \beta_4 (q_0 + pq_1)^2 (\ddot{q}_0 + p\ddot{q}_1)\Big] = 0 \end{aligned} \tag{11}$$

Collecting the terms of the same power of p in Eq. (11), we obtain a series of linear equations which the first equation is:

$$\ddot{q}_0(t) + \omega^2 q_0(t) = 0, \qquad q_0(0) = A, \quad \dot{q}_0(0) = 0 \tag{12}$$

with the solution:

$$q_0(t) = A\cos(\omega t). \tag{13}$$

Substitution of this result into the right-hand side of the second equation gives:

$$\begin{aligned} \ddot{q}_1(t) + \omega^2 q_1(t) &= \left(b_1 A - \frac{3}{4}c_1\beta_2 A^3 + 4c_1\beta_3 A + \frac{3}{4}c_1\beta_5 A^3\omega^2 - \frac{1}{4}c_1\beta_4 A^3\omega^2 + a_1 A\omega^2\right)\cos(\omega t) \\ &+ \frac{16}{3\pi}c_1\beta_3 A\cos(2\omega t) + \frac{1}{4}c_1 A^3\left(\beta_4\omega^2 + \beta_5\omega^2 - \beta_2\right)\cos(3\omega t). \end{aligned} \tag{14}$$

Solution of Eq. (14) should not contain the so-called secular term $\cos(\omega t)$. To ensure this, the right-hand side of this equation should not contain the terms *cos*, that is, the coefficients of *cos* must be zero:

$$\left(b_1 A - \frac{3}{4}c_1\beta_2 A^3 + 4c_1\beta_3 A + \frac{3}{4}c_1\beta_5 A^3\omega^2 - \frac{1}{4}c_1\beta_4 A^3\omega^2 + a_1 A\omega^2\right) = 0 \quad (15)$$

Eq. (10) for one term approximation of series with respect to p and for $p = 1$ yields:

$$a_1 = 0,\ b_1 = \omega^2 - \beta_1,\ c_1 = 1 \quad (16)$$

From Eqs. (15) and (16), we can easily find that the solution ω is:

$$\omega(A) = \sqrt{\frac{4\beta_1 + 3\beta_2 A^2}{4 + 3\beta_4 A^2 - \beta_3 A^2}} \quad (17)$$

2. Homotopy Analysis Method

Consider the nonlinear differential equation in general form:

$$N\left[Y(t)\right] = 0 \quad (18)$$

where N is a differential operator and $Y(t)$ is a solution. Applying the HAM to solve it, we first need to construct the following family of equations:

$$(1-q)\left\{L\left[\theta(t,q) - Y_0(t)\right]\right\} = h\,q\,N\left[\theta(t,q)\right] \quad (19)$$

where L is a properly selected auxiliary linear operator satisfying:

$$L(0) = 0 \quad (20)$$

$h \neq 0$ is an auxiliary parameter, and $Y_0(t)$ is an initial approximation. Obviously, when $q = 0$ Eq. (30) gives:

$$\theta(t,0) = Y_0(t). \quad (21)$$

Similarly, when $q = 1$, Eq. (19) is the same as Eq. (18) so that we have:

$$\theta(t,1) = Y(t). \quad (22)$$

Suppose that Eq. (19) has solution $\theta(t,q)$ that converges for all $0 \le q \le 1$ and for properly selected h and the auxiliary linear operator L. Suppose further that $\theta(t,q)$ is infinitely differentiable at $q = 0$, that is:

$$\left.\frac{\partial^k \theta(t,q)}{\partial q^k}\right|_{q=0}, \quad k = 1,2,3,... \quad (23)$$

exists. Thus, as q increases from 0 to 1, the solution $\theta(t,q)$ varies continuously from the initial approximation $Y_0(t)$ to the solution $Y(t)$ of the original Eq. (18). Clearly Eqs. (18) and (19) give an indirect relation between the initial approximation $Y_0(t)$ and the general solution $Y(t)$.

A direct relationship between the two solutions is described as follows. Consider the Maclaurin's series of $\theta(t,q)$ about q as:

$$\theta(t,q)=\theta(t,0)+\sum_{k=1}^{\infty}Y_k(t)q^k, \tag{24}$$

where:

$$Y_k(t)=\frac{1}{k!}\left.\frac{\partial^k\theta(t,q)}{\partial q^k}\right|_{q=0}. \tag{25}$$

Assume that the series (24) converges at $q=1$. From Eqs. (21), (22) and (24), we have the relationship:

$$Y(t)=Y_0(t)+\sum_{k=1}^{\infty}Y_k(t). \tag{26}$$

HAM provides a general approach to derive the governing equation of $Y_k(t)$. Substituting the series (24) into Eq. (19) and equating the coefficient of the like power of q, we get the *k*-th order deformation equations:

$$L\left[Y_k(t)-\chi_k Y_{k-1}(t)\right]=h R_k(t), \tag{27}$$

where:

$$R_k(t)=\frac{1}{(k-1)!}\left.\frac{d^{k-1}N\left[\theta(t,q)\right]}{dq^{k-1}}\right|_{q=0} \tag{28}$$

and

$$\chi_k=\begin{cases}0, & k\le 1.\\ 1, & k\le 2.\end{cases} \tag{29}$$

It is very important to emphasize that Eq. (27) is linear. If the first (*k*-1)-th order approximations have been obtained, then the right-hand side $R_k(t)$ will be obtained. The limit-cycles of oscillating systems are periodic motions with the period $T=2\pi/\omega$, And thus $y(\tau)$ can be expressed by such a set of base functions:

$$\left\{\cos(k\tau)\,|\,k=0,1,2,3,...\right\}, \tag{30}$$

that

$$y(\tau)=\sum_{k=0}^{\infty}\left(\beta_k\cos(k\tau)\right),\ y(0)=A,\ \dot{y}(0)=0 \tag{31}$$

where β_k are coefficients. It is obvious that:

$$y_0(\tau)=A\cos(\tau), \tag{32}$$

is a good initial guess of $y(\tau)$. To ensure this under the rule of solution expression described by (31), one chooses such an auxiliary linear operator:

$$L\left[\theta(\tau,q)\right]=\omega_0^2\left[\frac{\partial^2\theta(\tau,q)}{\partial\tau^2}+\theta(\tau,q)\right], \tag{33}$$

that

$$L\left[C\cos\tau\right]=0, \tag{34}$$

where C is coefficient. Then, due to Eq. (19), one defines the non-linear operator:

$$N\left[\theta(\tau,q),\Omega(q)\right]=\Omega^2(q)\frac{\partial^2\theta(\tau,q)}{\partial\tau^2}-F\left[\theta(\tau,q),\frac{\partial\theta(\tau,q)}{\partial\tau},\frac{\partial^2\theta(\tau,q)}{\partial\tau^2},\Omega(q)\right]. \tag{35}$$

Then, $\theta(\tau,q)$, $\Omega(q)$, can be expanded in the Maclaurin series of q as follows:

$$\theta(\tau,q)=\sum_{k=0}^{+\infty}y_k(\tau)q^k, \tag{36}$$

$$\Omega(q)=\sum_{k=0}^{+\infty}\omega_k\,q^k. \tag{37}$$

To ensure this, the right-hand side term $R_k(t)$ of (28) should not contain the terms cos, i. e. the coefficients of cos must be zero. So, rewrite:

$$R_k(\tau)=\sum_{m=1}^{\varphi(k)}\left(c_{k,m}\cos(m\tau)\right). \tag{38}$$

Then, one gains one algebraic equation as:

$$c_{k,1}\left(\omega_0,\omega_1,...,\omega_{k-1},A_0,A_1,A_2,...,A_{k-1}\right)=0, \tag{39}$$

which determine ω_{k-1} as a function of A_{k-1}. Under transformation $\tau=\omega t$ Eq. (4) becomes:

$$\omega^2q''+\beta_1q+\beta_2q^3+\beta_3\omega^2qq'^2+\beta_4\omega^2q^2q''=0. \tag{40}$$

Substituting Eq. (32) into Eq. (35), we get the algebraic Eq. (39) for this situation and this yields the following formula for nonlinear frequency, which is the same as Eq. (17):

$$\omega_0(A)=\frac{\sqrt{\left(4-A^2\beta_3+3A^2\beta_4\right)\left(3A^2\beta_2+4\beta_1\right)}}{4-A^2\beta_3+3A^2\beta_4}. \tag{41}$$

3. Bubnov-Galerkin Method

Perhaps the best known of the approximate methods is the Bubnov-Galerkin procedure. Consider the initial boundary value problem:

$$\ddot{q}+\omega^2 q+f\left(q,\dot{q},\ddot{q}\right)=0, \quad q(0)=A, \ \dot{q}(0)=0. \tag{42}$$

To develop this method, select a trial solution $q_0(t)$, which satisfies initial conditions only. In this method the weighting functions are chosen to be the basis functions of the trial solution, i. e.:

$$w(t)=q_0(t). \tag{43}$$

To select the optimal function for $\omega(A)$, the equation residual:

$$R_E\left(q_0(t)\right)=\ddot{q}_0+\omega^2 q_0+f\left(q_0,\dot{q}_0,\ddot{q}_0\right) \tag{44}$$

is created. If the trial function is the exact solution, then the residual is zero. According to Galerkin weighted residual procedure, weighted integrals of the equation residual R_E must be equal to zero, i. e.:

$$I(\omega)=\int_0^{2\pi/\omega} w(t)R_E\left(q_0(t)\right)dt=0. \tag{45}$$

By forming the equation residual, we have:

$$I(\omega)=\frac{A^2\pi}{4\omega}\left(\beta_3 A^2\omega^2+4\beta_1-4\omega^2-3A^2\beta_4\omega^2+3A^2\beta_2\right). \tag{46}$$

The solution of the Eq. (46) with respect to ω gives:

$$\omega(A)=\sqrt{\frac{4\beta_1+3\beta_2 A^2}{4+3\beta_4 A^2-\beta_3 A^2}}. \tag{47}$$

Discussion

To verify the accuracy of the obtained analytical solutions, the authors plot the analytical solutions and numerical results simultaneously.

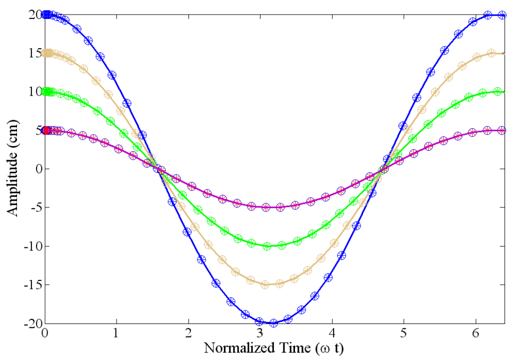

Fig. 2. Comparison of the results of analytical solutions with the numerical solution. Symbols: numerical solution; Solid line: analytical solutions

As can be observed in Fig. 2, the obtained first order approximation of $q(t)$ using five analytical methods shows an excellent agreement with numerical results using fourth-order Runge-Kutta method. The exact analytical solutions exhibit that the first term in series expansions is sufficient to achieve a highly accurate solution of the problem. Table 1 shows the values for nonlinear frequencies ω_0 as a function of amplitude A, for different values of system parameters.

Table 1. Nonlinear frequencies ω_0 as a function of amplitude A

System parameters				ω_0
A	$\beta_1 \times 10^3$	$\beta_2 \times 10^3$	$\beta_2 \times 10^3$	
1	65.5	214.4	4.6	256.0947033
2	41.9	137.2	4.6	204.8037687
3	2.62	2.143	1.149	51.20024855
4	5.9	4.82	1.149	76.80141328
5	94.4	308.7	4.6	307.2553308
10	128.45	420.2	4.6	358.6670051
20	132.94	434.88	4.6	365.6475255

Conclusion

In this research work modern powerful analytical methods were employed to solve the governing equation of nonlinear vibration of a cantilever beam. The analytical solutions yield a perceptive understanding of the effect of system parameters and initial conditions. The accuracy of the obtained analytical solutions is verified by numerical methods. The accuracy of the results demonstrates that these methods can be potentiality used for the analysis of strongly nonlinear oscillation problems.

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