570. Bifurcation Analysis and Rare Attractors in Driven Damped Pendulum Systems

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Abstract. The paper reports the complete bifurcation analysis of the driven damped pendulum systems by the new method of complete bifurcation groups (MCBG). Construction of complete bifurcation groups is based on the method of stable and unstable periodic regimes continuation on a parameter. Global bifurcation analysis of the driven damped pendulum systems allows determination of new bifurcation groups with rare attractors and chaotic regimes.

Keywords: pendulum systems, complete bifurcation analysis, method of complete bifurcation groups, rare attractors, chaos, domains of attraction.

Introduction

It is known that behavior of a periodically driven damped pendulum may be very complex and sometimes with the unexpected phenomena such as stable hilltop vibrations, complex subharmonical and quasi-periodical vibrations, and different rotations [1-7]. Recent efforts in nonlinear dynamics show that continuation on unstable branches of the periodic regimes allows finding new so-called rare attractors (RA), cannot be located systemically by other methods, and in particular for Duffing equation and other archetypical models [7-13]. Rare attractors are stable periodic regimes, which exist only in the relatively narrow intervals of the change of variable parameter. Rare attractors may be periodic, quasi-periodic or chaotic. They may belong to five different types such as tip, island, dumb-bell, hysteresis and protuberance. The systematic research of rare attractors is based on the recently developed method of complete bifurcation groups (MCBG), which allows conducting more complete global analysis of the dynamical systems. The method is based on the ideas of Poincaré, Birkhoff, Andronov and used together with Poincaré mappings, mappings from a line and from a contour, basins of attraction, etc. [7-13]. The main feature of the MCBG is that it uses nT-branches continuation without their break in bifurcation points and connected with protuberances born from some bifurcation points by period doubling (e.g. fold, pitchfork and other bifurcations). So, unstable periodic solutions, during branch continuation in single parameter space, are corner-stone meaning in the MCBG. New bifurcation sub-groups, obtained by the MCBG, such as complex protuberances and unstable periodic infinitiums (UPI) allow predicting and finding new (unknown) regular and chaotic rare attractors in the system [12]. In this paper we investigate global dynamics of a more common model than the models used in the usual previous investigations. Three cases are under consideration: a) with horizontal excitation of the vibrating support, b) with vertical excitation, c) with excitation under some angle α from the horizontal direction.

Our aim is building complete bifurcation diagrams and finding unknown rare regular and chaotic attractors using complete bifurcation analysis for such important parameters of the model: frequency of periodical excitation and angle α of direction of vibrating of the support.

The main results are presented by complete bifurcation diagrams for variable parameters of the system. We have revealed different new rare regular and chaotic attractors and some other new nonlinear phenomena such as coexisting of different types of periodic attractors (usual P1, hilltop P1, subharmonical solutions) and different types of subharmonical rotations, multiplicity of chaotic attractors and subharmonics. It is demonstrated that the direction α of the vibrating support is sufficient for global topology of the steady-state solutions of the system. Namely, some subharmonical isles with rare attractors exist only for a case, where the support is vibrating under some angle α. All results were obtained numerically, using software NLO [8] and SPRING [9], created in Riga Technical University.

Driven damped pendulum model

The studied dynamical model is presented in Fig. 1. The system has the harmonically vibrating point of suspension under some angle α from the horizontal direction to the vertical. Close models have been examined in some works [1-6]. The aim of our bifurcation analysis is to determine new rare attractors and new bifurcation groups. Some results of complete bifurcation analysis of the studied model have been examined in one of the previous works [14].

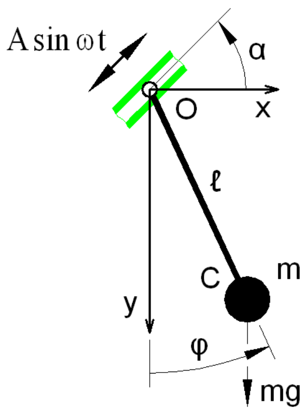

Fig. 1. The periodically driven damped pendulum with vibrating point of the support

The equation of motion for mathematical pendulum (Fig. 1) is as follows:

$$m\ell^2\ddot{\varphi} + b\dot{\varphi} + mg\ell\sin\pi\varphi + m\ell A\omega^2\sin\omega t\cos\pi(\varphi+\alpha) = 0\,, \tag{1}$$

where φ – angle of the pendulum, read-out from a vertical line; $\dot{\varphi}$ – angular velocity, where $\dot{\varphi} = d\varphi/dt$; α – angle of periodically excitation of the support, merged from the horizontal direction; t – time; m – mass, ℓ – length of the pendulum; g – gravitation constant; b – linear damping coefficient; A and ω – oscillation amplitude and frequency of the support.

For simplification we accept that the angle of complete rotation is not equal to 2π, but equals 2. The same for the horizontal excitation $\alpha = 0$ and for vertical excitation $\alpha = 0.5$. The variable parameters of the system are frequency ω of periodical excitation and angle α of direction of vibrating of the support. Other parameters are fixed: $m = 1$, $\ell = 1$, $b = 0.2$, $g = 9.81$, $A = 2$.

Results

Some results of bifurcation analysis of the model are presented in Figs. 2-5. In the first special case the symmetric model only with horizontal external force has 11 bifurcation groups (Fig. 2). In this figure stable solutions are plotted by solid lines and unstable – by thin lines (reddish online). One branch of P1 unstable hilltop near ω = 1.75 is not completed, because of problem of sufficiently high instability with maximal value less than 7200. Phase projections for four cross-sections of bifurcation diagram (see Fig. 2) are shown in Fig. 3: (a)-(c) for ω = 3.3, (d)-(e) for ω = 3.85, (f)-(i) for ω = 4.52, (j)-(l) for ω = 4.635. Using complete bifurcation analysis we have found different new rare regular and chaotic attractors, parameter regions with the coexistence of periodic attractors and subharmonical rotations.

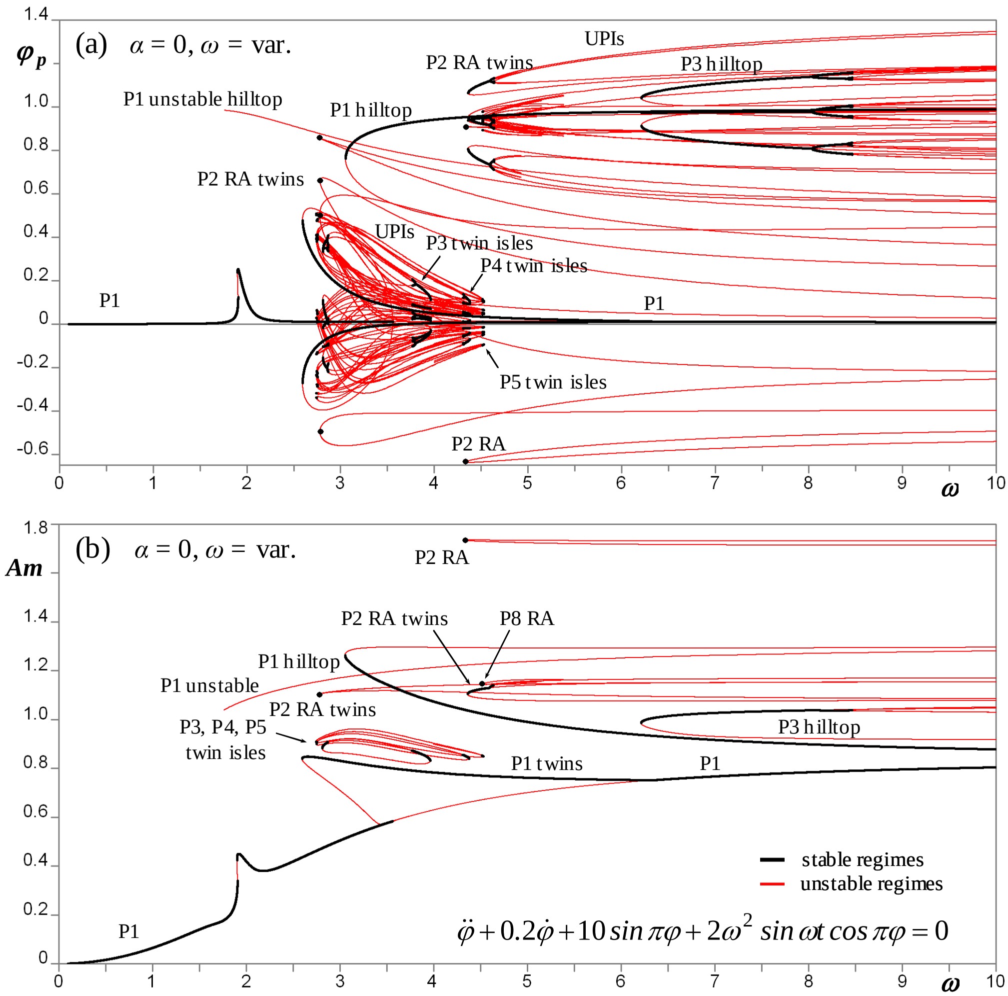

Fig. 2. The symmetric periodically driven damped pendulum (see Fig. 1) with the periodically vibrating point of suspension in horizontal direction. Bifurcation diagrams of the fixed periodic points of the angle of rotation φ_p and amplitude of rotation *Am* vs. oscillation frequency ω. Complete bifurcation diagrams of 11 bifurcation groups: three 1T, three 2T, two 3T, one 4T, one 5T and one 8T. There is submerged subharmonic twin isles 3T-5T form cluster. The system has many rare attractors of different kinds.

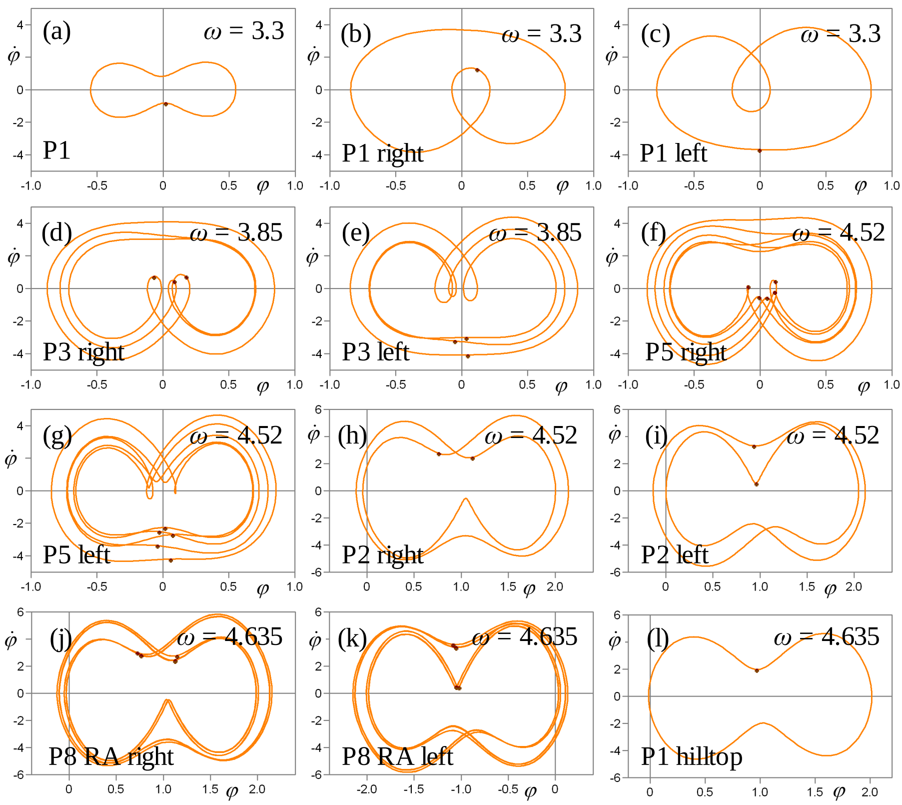

Fig. 3. Phase projections for four cross-sections of bifurcation diagram (see Fig.2): (a)-(c) for $\omega = 3.3$, (d)-(e) for $\omega = 3.85$, (f)-(i) for $\omega = 4.52$, (j)-(l) for $\omega = 4.635$.

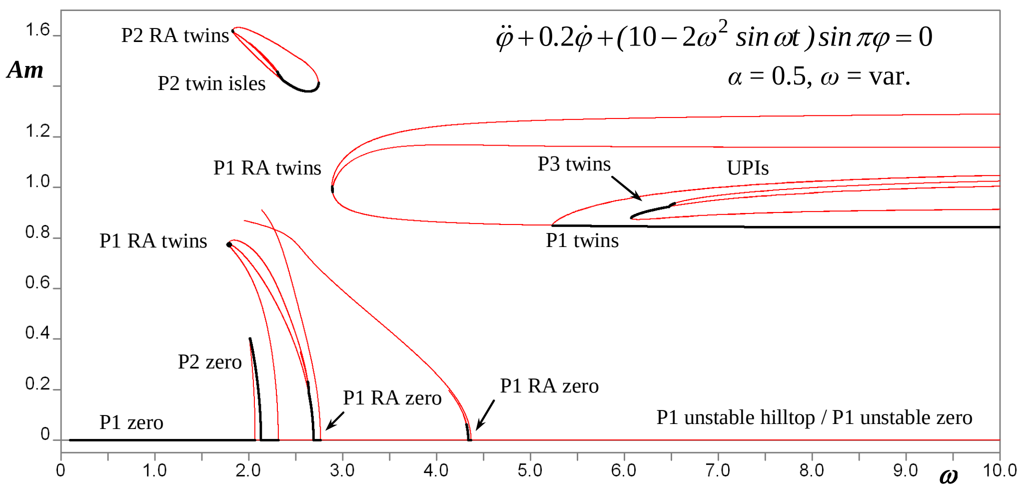

Fig. 4. The symmetric pendulum with the periodically vibrating point of suspension in vertical direction. Bifurcation diagrams of the fixed periodic points of the angle of rotation φ_p and amplitude of rotation *Am* vs. oscillation frequency ω. Complete bifurcation diagrams of 5 bifurcation groups: three 1T, one 2T and one 3T.

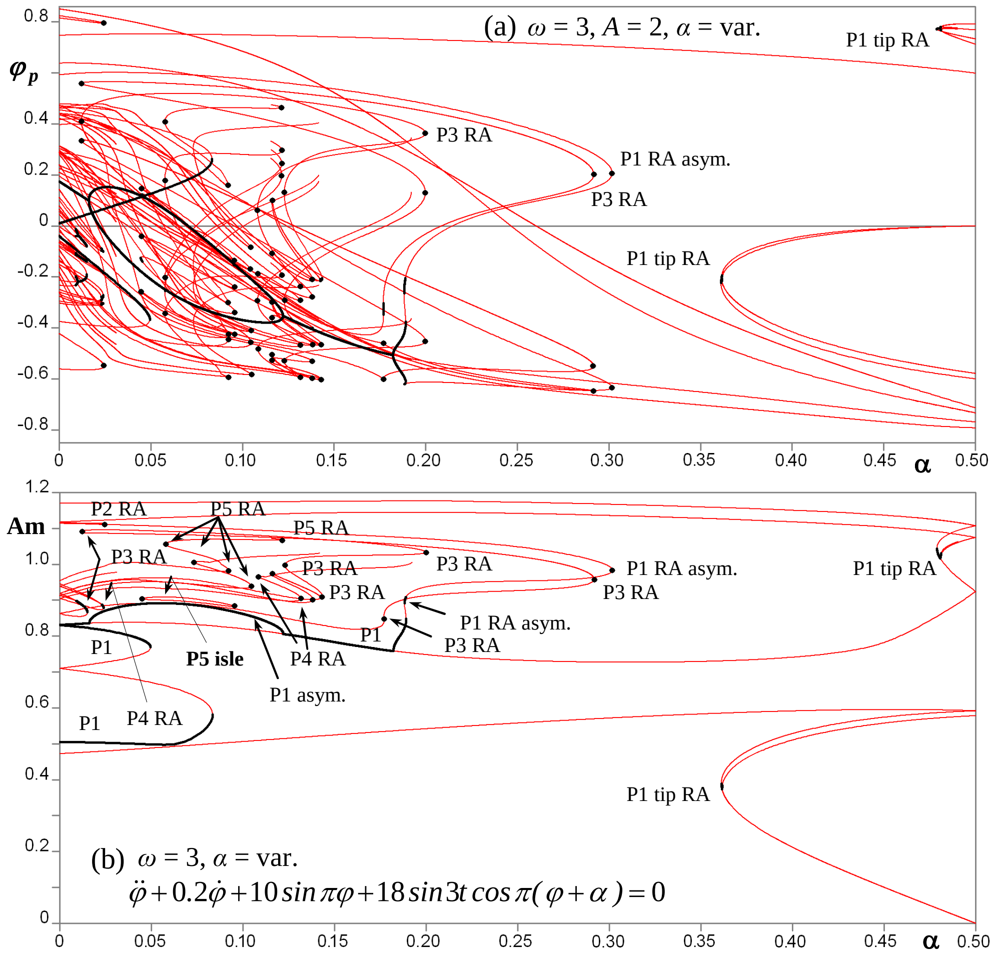

Fig. 5. The periodically driven damped pendulum (see Fig. 1) with the periodically vibrating point of suspension under angle. Bifurcation diagrams of the fixed periodic points of the angle of rotation φ_p and amplitude of rotation *Am* vs. angle α of periodic excitation of the support. The system has many rare attractors (RA) marked by black circles. Inside the each marked circle there are rare bifurcation sub-groups with period-doubling cascade and rare chaotic attractors.

In the second case the symmetric model only with vertical external force has 5 bifurcation groups (Fig. 4): three 1T, one 2T and one 3T. Two branches of P1 unstable zero twins near ω = 1.96 and ω = 2.13 are not completed, because of problem of sufficiently high instability with maximal value less than 2600.

The results obtained by the method of complete bifurcation groups indicate that there are stable hilltop solutions not only period-1, but more complicated subharmonic hilltop attractors, e.g. period-3, and hilltop chaotic attractors (see Fig. 4). All bifurcation groups with rare attractors are of a tip kind and each of them has not only periodic attractors, but chaotic attractors as well. Fig. 5 demonstrates that the direction α of the vibrating support is sufficient for global topology of the steady-state solutions of the system. Namely, some subharmonical isles with rare attractors exist only for a case, where the support is vibrating under some angle α from 0.044 to 0.095.

Conclusions

It is demonstrated that application of the method of complete bifurcation groups allows conducting global bifurcation analysis of the periodically driven damped pendulum with the vibrating point of the support under some angle α from the horizontal direction to vertical direction, and finding new bifurcation groups with rare attractors and chaotic regimes. Some of the established new effects can be employed for the parametric stabilization of unstable oscillations in technological processes or for other applications in the mechanics, mechanical engineering and other nonlinear dynamic systems. Authors hope to attract attention of scientists and engineers to solving important problems of nonlinear oscillations analysis by the MCBG. Rare dangerous or rare useful attractors may find application not only in the pendulum-like dynamical systems, but also in social sciences, biology and medicine.

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